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ABSTRACT

Unit 13 in the SMSG secondary school mathematics series is a student text covering the following topics in geometry: common sense and organized knowledge; sets, real numbers and lines; lines, planes, and separation; angles and triangles; congruences; proof; geometric inequalities; perpendicular lines and planes in space; parallel lines in a plane; and parallels in space. Appendices cover symbols, postulates of addition and multiplication, rational and irrational numbers, square roots, how to draw figures in 3-space, and proof of theorems on perpendicularity. (DT)

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STUDENT'S TEXT UNIT NO. 13

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SCHOOL MATHEMATICS STUDY GROUP

School Mathematics Study Group

Geometry

Unit 13

Geometry

Student's Text, Part I

REVISED EDITION

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FOREWORD

The increasing contribution of mathematics to the culture of the modern world, as well as its importance as a vital part of scientific and humanistic education, has made it essential that the mathematics in our schools be both well selected and well taught.

With this in mind, the various mathematical organizations in the United States cooperated in the formation of the School Mathematics Study Group (SMSG). SMSG includes college and university mathematicians, teachers of mathematics at all levels, experts in education, and representatives of science and technology. The general objective of SMSG is the improvement of the teaching of mathematics in the schools of this country. The National Science Foundation has provided substantial funds for the support of this endeavor.

One of the prerequisites for the improvement of the teaching of mathematics in our schools is an improved curriculum--one which takes account of the increasing use of mathematics in science and technology and in other areas of knowledge and at the same time one which reflects recent advances in mathematics itself. One of the first projects undertaken by SMSG was to enlist a group of outstanding mathematicians and mathematics teachers to prepare a series of textbooks which would illustrate such an improved curriculum.

The professional mathematicians in SMSG believe that the mathematics presented in this text is valuable for all well-educated citizens in our society to know and that it is important for the precollege student to learn in preparation for advanced work in the field. At the same time, teachers in SMSG believe that it is presented in such a form that it can be readily grasped by students.

In most instances the material will have a familiar note, but the presentation and the point of view will be different. Some material will be entirely new to the traditional curriculum. This is as it should be, for mathematics is a living and an ever-growing subject, and not a dead and frozen product of antiquity. This healthy fusion of the old and the new should lead students to a better understanding of the basic concepts and structure of mathematics and provide a firmer foundation for understanding and use of mathematics in a scientific society.

It is not intended that this book be regarded as the only definitive way of presenting good mathematics to students at this level. Instead, it should be thought of as a sample of the kind of improved curriculum that we need and as a source of suggestions for the authors of commercial textbooks. It is sincerely hoped that these texts will lead the way toward inspiring a more meaningful teaching of Mathematics, the Queen and Servant of the Sciences.

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PREFACE

This book is designed for the one-year introductory course in geometry which is usually taught in the tenth grade. Students in this grade normally have learned a fair amount of informal geometry, including the calculation of areas and volumes for various elementary figures, the Pythagorean relation, and the use of similar right triangles to calculate unknown heights and distances. Students who have not learned this material may have to be given some extra attention, but the book should still be teachable, at a suitably adjusted pace. In algebra, no special preparation is required beyond the knowledge and skills normally acquired in the ninth grade.

The book is devoted mainly to plane geometry, with a few chapters on solid geometry, and a short introduction to analytic geometry at the end. It seems natural, in a preface, to give an account of the novel features in the treatment. We are aware, of course, of a danger in so doing. A long string of novelties, offered for the reader's special attention, may very well convey the impression that the authors have been engaged in an unhealthy pursuit of innovation for its own sake. This is by no means the way in which we have conceived our task. We began and ended our work with the conviction that the traditional content of Euclidean geometry amply deserves the prominent place which it now holds in high-school study; and we have made changes only when the need for them appeared to be compelling.

The basic scheme in the postulates is that of G. D. Birkhoff. In this scheme, it is assumed that the real numbers are known, and they are used freely for measuring both distances and angles. This has two main advantages.

In the first place, the real numbers give us a sort of head start. It has been correctly pointed out that Euclid's postulates are not logically sufficient for geometry, and that the treatments based on them do not meet modern standards of rigor. They were improved and sharpened by Hilbert. But the foundations of geometry, in the sense of Hilbert, are not a part of elementary mathematics, and do not belong in the tenth-grade curriculum. If we assume the real numbers, as in the Birkhoff treatment, then the handling of our postulates becomes a much easier task, and we need not face a cruel choice between mathematical accuracy and intelligibility.

In the second place, it seems a good idea in itself to connect up geometry with algebra at every reasonable opportunity, so that knowledge in one of these fields will make its natural contribution to the understanding of both. Some of the topics usually studied in geometry are essentially algebraic. This is true, for example, of the proportionality relations for similar triangles. In this book, such topics are treated algebraically, so as to bring out the connections with the work of the ninth and eleventh grades.

We hope that the statements of definitions and theorems are exact; we have tried hard to make them so. Just as a lawyer needs to learn to draw up contracts that say what they are supposed to say, so a mathematics student needs to learn to write mathematical statements that can be taken literally. But we are not under the illusion that this sort of exactitude is a substitute for intuitive insight. We have, therefore, based the design of both the text and the problems on our belief that intuition and logic should move forward hand in hand.

Chapter 1

COMMON SENSE AND ORGANIZED KNOWLEDGE

1-1. Two Types of Problems.

Consider the following problems:

1. A line segment 14 inches long is broken into two segments. If one of the two smaller segments is 6 inches long, how long is the other one?

2. In a certain rectangle, the sum of the length and the width is 14 (measured in inches). A second rectangle is three times as long as the first, and twice as wide. The perimeter of the second rectangle is 72. What are the dimensions of the first rectangle?

The answer to Problem 1, of course, is 8 inches, because $6 + 8 = 14$. We could solve this problem algebraically, if we wanted to, by setting up the equation

$$6 + x = 14,$$

and solving to get $x = 8$. But this seems a little silly, because it is so unnecessary. If all algebraic equations were as superfluous as this one, then no serious-minded person would pay any attention to them; in fact, they would probably never have been invented.

Problem 2, however, is quite another matter. If the length and width of the first rectangle are x and y, then the length and width of the second rectangle are $3x$ and $2y$. Therefore,

$$3x + 2y = \frac{1}{2} \cdot 72 = 36$$

because the sum of the length and width is half the perimeter. We already know that $x + y = 14$. Thus we have a system of two linear equations in two unknowns:

$$\begin{aligned} x + y &= 14 \\ 3x + 2y &= 36. \end{aligned}$$

To solve, we multiply each term in the first equation by 2, getting

$$2x + 2y = 28,$$

and then we subtract this last equation, term by term, from the second. This gives

$$x = 8.$$

Since $x + y = 14$, we have $y = 6$, which completes the solution of our problem. It is easy to check that a length of 8 and a width of 6 satisfy the conditions of the problem.

In a way, these two problems may seem similar. But in a very important sense, they are different. The first is what you might call a common-sense problem. It is very easy to guess what the answer ought to be, and it is also very easy to check that the natural guess is actually the right answer. The second problem is entirely another matter. To solve the second problem, we need to know something about mathematical methods.

There are cases of this kind in geometry. Consider the following statements:

1. If a triangle has sides of length 3, 4 and 5, then it is a right triangle, with a right angle opposite the longest side.

2. Let a triangle be given, with sides a, b and c. If

$$a^2 + b^2 = c^2,$$

then the triangle is a right triangle, with a right angle opposite the longest side.

The first of these facts was known to the ancient Egyptians. They checked it by experiment. You can check it yourself, with a ruler and compass, by drawing a 3-4-5 triangle, and then measuring the angle opposite the longest side with a protractor. You should bear in mind, of course, that this check is only approximate. For example, if the angle were really $89^\circ 59' 59''$, instead of 90° exactly, you would hardly expect to tell the difference by drawing your figure and then taking a reading with your protractor. Nevertheless, the "Egyptian method" is a sound common-sense method of verifying an experimental fact.

The Egyptians were extremely skillful at making physical measurements. The edges of the base of the great pyramid are about 756 feet long; and the lengths of these four edges agree, with an error of only about two-thirds of an inch. Nobody seems to know, today, how the builders got such accuracy.

Statement 2 above was not known to the Egyptians; it was discovered later, by the Greeks. This second statement is very different from the first. The most important difference is that there are infinitely many possibilities for a, b and c. For instance, you would have to construct triangles, and take readings with a protractor, for all of the following cases,

a	b	c
1	1	$\sqrt{2}$
2	1	$\sqrt{5}$
2	2	$\sqrt{8}$
3	1	$\sqrt{10}$
3	2	$\sqrt{13}$
3	3	$\sqrt{18}$

and so on, endlessly. It seems pretty hopeless to try to verify our general statement by experiment, even approximately. Therefore, a reasonable person would not be convinced that Statement 2 was true in all cases until he had seen some logical reason why it should be true in all cases.

In fact, this is why it was the Greeks, and not the Egyptians, who discovered that our second statement is true. The Egyptians had lots of common-sense knowledge of geometry. But the Greeks found something better, and much more powerful: they discovered the science of exact geometrical reasoning. By exact reasoning, they learned a great deal that had not been known before their time. The things that they learned were the first big step toward modern mathematics, and hence, toward modern science in general.

Problem Set 1-1

1. Try the following experiment. Take a piece of string, about six feet long, and put it on the floor in the form of a loop with the ends free:

Then pull the ends of the string apart, making the loop gradually smaller, and stop when you think that the loop is the size of your own waist. Then check the accuracy of your guess by wrapping the string around your waist. After you have checked, read the remarks at the end of this set of problems.

2. In this pair of questions, the first can be answered by "common sense." State only its answer. The second requires some arithmetic or algebraic process for its solution. Show your work for it.
   a. What is half of 2?
   b. What is half of 135,790?
3. Answer as in Problem 2:
   a. One-third of the distance between two cities is 10 miles. What is the entire distance?
   b. The distance between two cities is 7 miles more than one-third the distance between them. What is the distance between them?

*4. Answer as in Problem 2:

a. If a 5-inch piece of wire is cut into two parts so that one part is 4 times as long as the other, what are the lengths of the parts?

b. If a 5-inch piece of wire is cut into two parts such that a square formed by bending one piece will have four times the area of a square formed by bending the other, what are the lengths of the parts?

5. If the sides of a triangle are 5, 12, 13, is it a right triangle?

6. If two students carefully and independently measure the width of a classroom with rulers, one measuring from left to right and the other from right to left, they are likely to get different answers. You may check this with an experiment. Which of the following are plausible reasons for this?

a. The rulers have different lengths.

b. One person may have lost count of the number of feet in the width.

c. Things are longer (or shorter) from left to right than right to left.

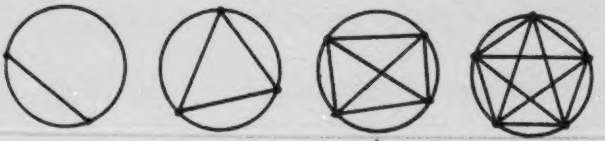
d. The errors made in changing the position of the ruler accumulate, and the sum of the small errors makes a discernable error.

7. Show that $n^2 - 2n + 2 = n$ if $n = 1$. Is the equation true when $n = 2$? Is it true when $n = 3$? Is it always true?

8. a. If 3^2, 5^2 and 7^2 are divided by 4, what is the remainder in each case?

b. How many odd integers would you have to square and divide by 4 to guarantee that the remainder would always be the same?

9.

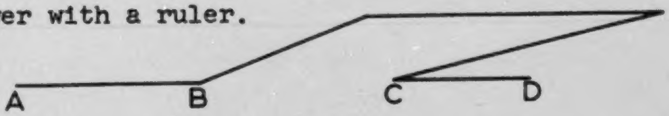


Number of points connected	2	3	4	5	6
Number of regions formed	2	4	8	16	?

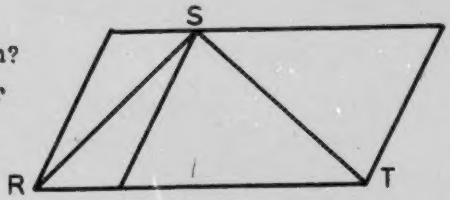
Replace the question mark by the number you think belongs there. Verify your answer by making a drawing in which six points on a circle are connected in all possible ways.

10. The following optical illusions show that you cannot always trust appearances. "Things are seldom what they seem; skim milk masquerades as cream." From "H.M.S. Pinafore" by Gilbert and Sullivan.

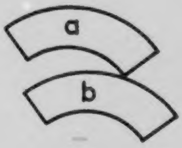
a. Is CD a continuation of AB?
Test your answer with a ruler.



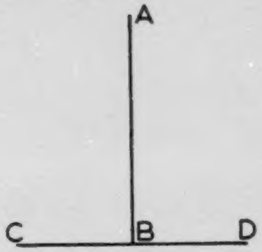
b. Are RS and ST equal in length? Compare the lengths with your ruler or compass.



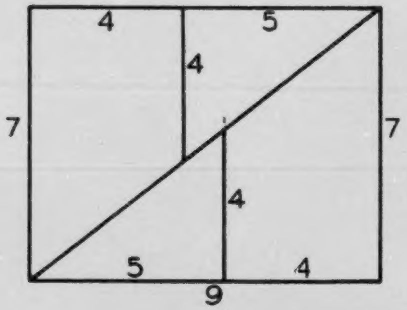
c. Which figure has the greater area?



d. Which is longer, AB or CD?
Check with your ruler.



*11. Use a ruler to check the accuracy of the measurements of the figure. Show that if these measurements are correct the sum of the areas of the four pieces of the rectangle is more than the area of the rectangle. Odd, isn't it?



*12. A trip of 60 miles is to be done at an average speed of 60 m.p.h. The first 30 miles are done at 30 m.p.h. At what rate must the remaining 30 miles be covered?

Remarks on Problem 1. Nearly everybody makes a loop about twice as big as it should be. You can get much better results by the following method. The circumference of a circle is equal to π times the diameter, and π is approximately equal to 3. Therefore, the diameter is about one-third of the circumference.

If your waist measure is, say, 21 inches, this means that the loop on the floor should be about 7 inches across. This will look unbelievably small, but if you have thought the problem out mathematically, you will have the courage of your convictions.

This is one of a large number of cases in which even a very crude mathematical approach to a problem is better than an outright leap in the dark.

1-2. An Organized Logical Development of Geometry.

If you stop to think, you will realize that by now you know a great many geometrical facts. For example, you know how to find the area of a rectangle, and of a right triangle, and perhaps of a triangle in general, and you know the Pythagorean relation for right triangles. Some of the things that you know are so simple and obvious that it might never occur to you to even put them into words, let alone to wonder whether or why they are true. The following is a statement of this type:

Two straight lines cannot cross each other in more than one point.

But some of them, like the Pythagorean relation, are not obvious at all, but rather surprising. We would like to organize our knowledge of geometry, in an orderly way, in such a way that these more complicated statements can be derived from simple statements. This suggests that we ought to be able to make a list of the facts of geometry, with the simplest and easiest statements coming first, and the hard ones coming later. We might try to arrange the statements in such an order that each statement in the list can be derived from the preceding statements by logical reasoning.

Actually, we shall carry out a program that is very much like this. We will state definitions, as clearly and exactly as we can; and we will establish the facts of geometry by giving logical proofs.

The statements that we prove will be called *theorems*. (*The proving of theorems is not a spectator-sport, any more than arithmetic is: the best way to learn about it is by doing it*. Therefore, in this course, you will have lots of opportunities to prove lots of theorems for yourself.)

While nearly all of the statements that we make about geometry are going to be proved, there will be some exceptions. The simplest and most fundamental statements will be given without proofs. These statements will be called *postulates*, and will form the foundation on which we will build. In the same way, we will use the simplest and most fundamental terms of geometry without defining them; these will be called the *undefined terms*. The definitions of the other terms that we use will be based on them.

At first glance, it might seem better to define every term that we use, and to prove every statement that we make. With a little reflection, we can convince ourselves that this can't be done.

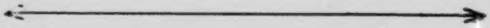
Consider first the question of the postulates. Most of the time, when we prove a theorem, we do so by showing that it follows logically from theorems that have already been proved. But it is clear that proofs of theorems cannot always work this way. In particular, the *first* theorem that we prove cannot possibly be proved this way, because in this case there aren't any previously proved theorems. But we have to start somewhere. This means that we have to accept some statements without proof. These unproved statements are the postulates.

The purpose of stating postulates is to make it clear just where we are starting, and just what sort of mathematical objects we are studying. We can then build up a solid, organized body of facts about these mathematical objects.

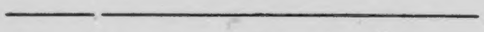
Just as we start with some unproved statements, so we start with some undefined terms. Most of the time, when we give a definition of a new geometric term, we define it by means of other geometric terms which have already been defined. But it is clear that definitions cannot always work this way. In particular, the *first* definition that we state cannot possibly be stated in this way, because in this case there aren't any previously defined geometric terms. But we have to start somewhere. This means that we introduce some geometric terms without defining them, and then use these basic terms in our first definitions. We shall use the simplest and most fundamental geometric terms without making any attempt to give definitions for them. Three fundamental *undefined terms* will be *point*, *line* and *plane*.

Postulates, of course, are not made up at random. (If they were, geometry would be of no interest or importance.) Postulates describe fundamental properties of space. In the same way, the undefined terms *point*, *line* and *plane* are suggested by physical objects. To get a reasonably good picture of a point, you make a dot on paper with a pencil. To get a better approximation of the mathematical idea of a point, you should first sharpen your pencil. The picture is still approximate, of course: a dot on paper must cover *some* area, or you couldn't see it at all. But if you think of dots made by sharper and sharper pencils, you will have a good idea of what we are driving at when we use the undefined term, *point*.

When we use the term *line*, we have in mind the idea of a *straight* line. A straight line, however, is supposed to extend infinitely far in both directions. Usually, we shall indicate this in pictures by arrowheads at the ends of the portion of the line that we draw, like this:



We shall have another term, *segment*, for a figure that looks like this:



A thin, tightly stretched string is a good approximation of a segment. An even thinner and more tightly stretched string is a better approximation. And so on.

Think of a perfectly flat surface, extending infinitely far in every direction, and you have a good idea of a plane.

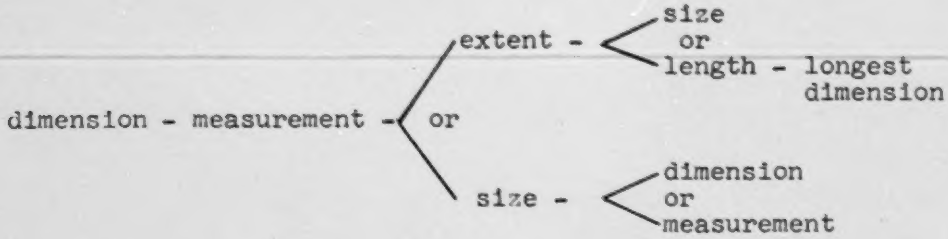
You should remember that none of the above statements are definitions. They are merely explanations of the ideas that people had in the back of their minds when they wrote the postulates. When we are writing proofs the information that we claim to have about points, lines and planes will be the information given by the postulates.

We have said that theorems are going to be proved by logical reasoning. We have not explained what logical reasoning is, and in fact, we don't know how to explain this in advance. As the course proceeds, you will get a better and better idea of what logical reasoning is, by seeing it used, and best of all by using it yourself. This is the way that all mathematicians have learned to tell what is a proof and what isn't.

At the beginning of the next chapter, we shall give a short account of the idea of a set, and a short review of the fundamentals of algebra for real numbers. Sets and algebra will be used throughout this course, and our study of geometry will largely be based on them. We shall think of them, however, as things that we are working with. They will not be an actual part of our system of postulates and theorems. They are supposed to be available at the start; some of our postulates will involve real numbers; and elementary algebra will be used in proofs. In fact, geometry and algebra are very closely connected, and both of them are easier to learn if the connections between them are brought out as soon as possible.

Problem Set 1-2

1. A student wanting to find the meaning of the word "dimension" went to a dictionary. This dictionary did not give definitions as we have them in geometry but did give synonyms of words. He made the following chart.



   a. Point out from the above chart a circular list of three terms each of which has its following term as a synonym. (In a circular list, the first term is assumed to follow the last.)

   b. Make a circular list which contains four such terms.

*2. Make a chart similar to that in Problem 1, starting with some word in your dictionary.

3. John convinced his mother that he did not track mud onto the living room rug by pointing out that it did not start raining until 5 o'clock and that he had been in his room studying since 4:30. He mentioned that a person cannot do something if he is not there. The thing he was proving (that he did not track mud) might be regarded as a theorem and the statement about a person not being able to do something if he is not there might be regarded as a postulate. Make another example of such a convincing argument and point out what corresponds to the theorem, what to the proof, and what to postulates.

4. Janie: What's an architect?
Mother: An architect? An architect is a man who designs buildings.
Janie: What's "designs"?
Mother: Well--plans.
Janie: Like we plan a picnic?
Mother: Yes, quite like that.
Janie: What are buildings?
Mother: Oh, Janie, you know -- houses, churches, schools.
Janie: Yes, I see.

Consider the above discussion. What were basic undefined terms as far as Janie was concerned?

5. The Stuarts have three children. Joe is a senior in high school. Karen is a seventh grader, and Beth is four. At the dinner table:

Joe: We learned a funny new word in geometry class today -- parallelepiped.
Karen: What in the world is it?
Joe: Well, it's a solid. You know what I mean by a solid figure -- it takes up some space. And it's bounded by planes. You know what a plane is, don't you?
Beth: Like a windowplane?
Joe: The word is a windowpane, but that's the idea. A parallelepiped is a solid bounded by parallelograms. A candy box is one, but it's a special one because the six faces are all rectangles. If you had a candy box and could shove it at one corner you'd get a parallelepiped. Got the idea?

In the above discussion what basic, undefined terms did Joe use in his description?

6. What do you think is wrong with the following faulty definitions?
   a. A square is something that is not round.
   b. A right triangle is a triangle each of whose angles has a measure of 90°.
   c. An equilateral triangle is when a triangle has three sides the same length.
   d. The perimeter of a rectangle is where you find the sum of the lengths of the sides of the rectangle.
   e. The circumference of a circle is found by multiplying the diameter by π.

*7. Indicate whether the following are true or false:
   a. It is possible to define each geometric term by using simpler geometric terms.
   b. Exact geometric reasoning leads us to geometric truths that cannot be deduced from measurement.
   c. Theorems are proved only on the basis of definitions and undefined terms.
   d. If you are willing to write in all the steps, each theorem can be deduced from postulates without making recourse to previous theorems.

Chapter 2

SETS, REAL NUMBERS AND LINES

2-1. Sets

You may not have heard the word set used in mathematics before, but the idea is a very familiar one. Your family is a set of people, consisting of you, your parents, and your brothers and sisters (if any). These people are the members of the set. Your geometry class is a set of students; its members are you and your classmates. A school athletic team is a set of students. A member of a set is said to belong to the set. For example, you belong to your family and to your geometry class, and so on. The members of a set are often called its elements; the two terms, members and elements, mean exactly the same thing. We say that a set contains each of its elements. For example, both your family and your geometry class contain you. If one set contains every element of another set, then we say that the first set contains the second, and we say that the second set is a subset of the first. For example, the student body of your school contains your geometry class, and your geometry class is a subset of the student body. We say that the subset lies in the set that contains it. For example, the set of all violinists lies in the set of all musicians.

Throughout this book, lines and planes will be regarded as sets of points. In fact, all the geometric figures that we talk about are sets of points. (You may regard this, if you like, as a postulate.)

When we say that two sets are equal, or when we write an equality $A = B$ between two sets A and B, we mean merely that the two sets have exactly the same elements. For example, let A be the set of all whole numbers between $\frac{1}{2}$ and $5\frac{1}{2}$, and let B be the set of all whole numbers between $\frac{1}{3}$ and $5\frac{1}{3}$. Then $A = B$, because each of the sets A and B has precisely the elements 1, 2, 3, 4 and 5. In fact, it very often happens that the same set can be described in two different ways; and if the descriptions look different, this doesn't necessarily mean that the sets are different.

Two sets intersect if there are one or more elements that belong to both of them. For example, your family and your geometry class must intersect, because you yourself belong to both of them. But two different classes meeting at the same hour do not intersect. The intersection of two sets is the set of all objects that belong to both of them. For example, the intersection of the set of all men and the set of all musicians is the set of all men musicians.

Passing to mathematical topics, we see that the set of all odd numbers is the set whose members are

$$1, 3, 5, 7, 9, 11, 13, 15, \ldots$$

and so on. The set of all multiples of 3 is the set whose members are

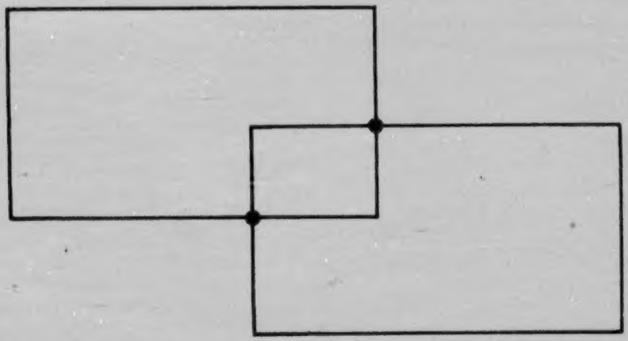
$$3, 6, 9, 12, 15, \ldots$$

and so on. The intersection of these two sets is

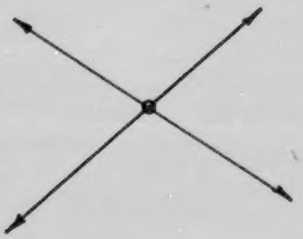
$$3, 9, 15, 21, \ldots$$

and so on; its members are the odd multiples of 3.

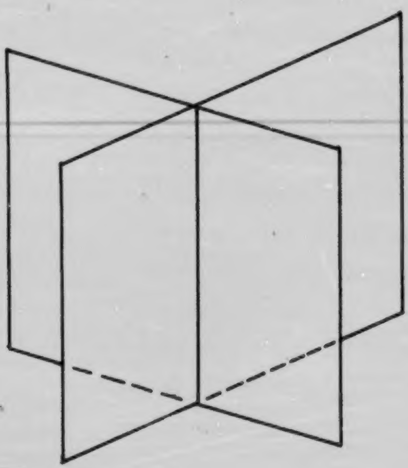
In the figure below, each of the two rectangles is a set of points, and their intersection contains exactly two points.



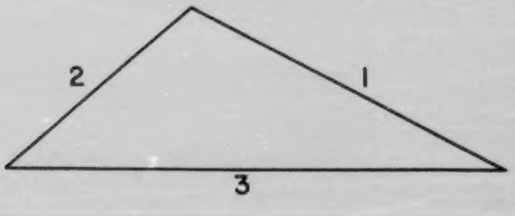
Similarly, each of the corresponding rectangular regions is a set of points, and their intersection is the small rectangular region in the middle of the figure. In the next figure, each of the two lines is a set of points, and their intersection consists of a single point:



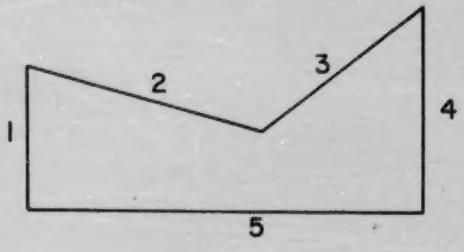
Below, we see two sets of points, each of which is a flat rectangular surface. The intersection of these two sets of points is a part of a straight line.



The union of the two sets is the set of all objects that belong to one or both of them. For example, the union of the set of all men and the set of all women is the set of all adults. The intersection, or the union, of three or more sets is defined similarly. Thus a triangle is the union of three sets, each of which is a subset of a line.



The figure below is the union of five sets, each of which is a subset of a line.



In some situations, it is convenient to use the idea of the empty set. The empty set is the set that has no members at all. This idea may seem a little peculiar at first, but it is really very much like the idea of the number 0. For example, the following three statements all say the same thing:

(1) There are no married bachelors in the world.

(2) The number of married bachelors in the world is zero.

(3) The set of all married bachelors in the world is the empty set.

Once we have introduced the empty set, then we can speak of the intersection of any two sets, remembering that the intersection may turn out to be the empty set.

For example, the intersection of the set of all odd numbers and the set of all even numbers is the empty set.

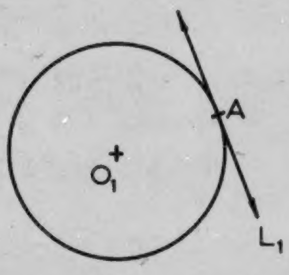
A word of warning: If you compare the definitions of the terms intersect and intersection, you will see that these two terms are not related in quite the simple way that you might expect. When we speak of the intersection of two sets, we allow the possibility that the intersection may be empty. But if we say that the two sets *intersect*, this always means that they have an element in common.

Another word of warning: Statements (2) and (3) above mean the same thing. But this does not mean that a set that contains only the number 0 is empty. For example, the equation $x + 3 = 3$ has 0 as its only root, and so the set of roots is not the empty set; the set of roots has exactly one element, namely, the number 0. On the other hand, the set of all roots of the equation $x + 3 = x + 4$ really is the empty set, because the equation $x + 3 = x + 4$ has no roots at all.

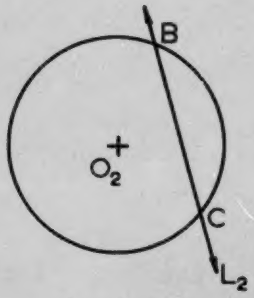
Problem Set 2-1

1. Let A be the set $\{3, 5, 6, 9, 11, 12\}$ (that is, the set whose members are 3, 5, 6, 9, 11, 12) and B be the set $\{4, 5, 7, 9, 10, 11\}$.
   What is the intersection of sets A and B? What is the union of A and B?
2. Consider the following sets:
   S_1 is the set of all students in your school.
   S_2 is the set of all boys in your student body.
   S_3 is the set of all girls in your student body.
   S_4 is the set of all members of the faculty of your school.
   S_5 is the set whose only member is yourself, a student in your school.
   a. Which *pairs* of sets intersect?
   b. Which set is the union of S_2 and S_3?
   c. Which set is the union of S_1 and S_5?
   d. Describe the union of S_1 and S_4.
   e. Which of the sets are sub-sets of S_1?

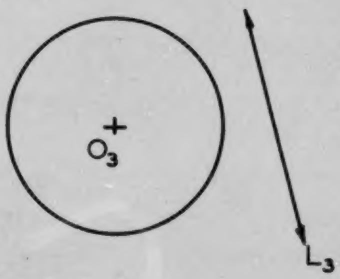
3. In the following figures, consider the line and the circle as two sets of points. In each case, what is their intersection?



Case I.

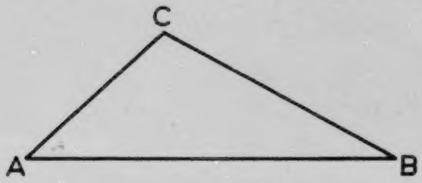


Case II.



Case III.

4. Consider a set of three boys, {A, B, C}. Any set of boys selected from these three will be called a committee.
   a. How many different two-member committees can be formed from the three boys?
   b. Show that any two of the committees in (a) intersect. What does the word "intersect" mean?
5. Consider the set of all positive even integers and the set of all positive odd integers. Describe the set which is the union of these two sets.
6. Describe the intersection of the two sets given in Problem 5.
7. In the figure, what is the intersection of the triangle ABC and the segment BC? What is their union?



8. Let A be the set of pairs of numbers (m,n) which satisfy the equation $4m + n = 9$.
   Let B be the set of pairs of numbers (m,n) which satisfy the equation $2m + n = 5$.
   Find the intersection of the sets A and B.

9. Let A be the set of pairs (x,y) for which $x + y = 7$.
Let B be the set of pairs (x,y) for which $x - y = 1$.
What is the intersection of A and B?

10. Let A be the set of pairs (x,y) for which $x + y = 3$.
Let B be the set of pairs (x,y) for which $2x + 2y = 7$.
What is the intersection of A and B?

11. Consider the set of all positive integers divisible by 2.
Consider the set of all positive integers divisible by 3.
    a. Describe the intersection of these two sets. Give its first four members.
    b. Write an algebraic expression for the intersection.
    c. Describe the union of the two sets. Give its first eight members.

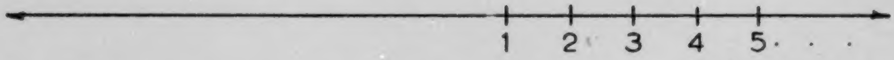
12. a. How many straight lines can be drawn through 2 points?
    b. If three points do not lie in a straight line, how many straight lines can be drawn through pairs of the points?
    c. If four points are given and no set of three of them lie in a straight line, how many straight lines can be drawn containing sets of two of the points? Answer the same question if five points are given.
    *d. Answer Question c if n points are given.

2-2. The Real Numbers

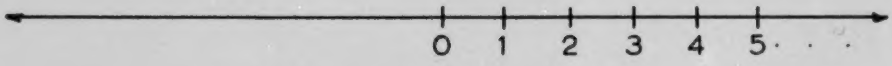
The first numbers that you learned about were the "counting numbers" or "natural numbers",

$$1, 2, 3, 4, 5, \ldots$$

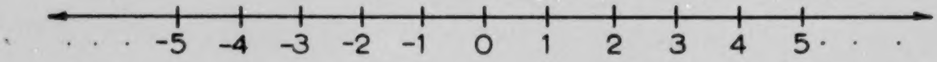
and so on. (You knew about these before you learned to read or write. And ancient man learned to count long before the invention of writing.) The counting numbers never end, because starting with any one of them, we can always add 1, and get another one. We may think of the counting numbers as arranged on a line, starting at some point and continuing to the right, like this:



To the left of 1, we put in the number 0, like this:

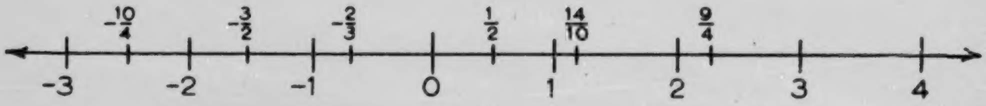


And the next step is to put in the negative whole numbers, like this:



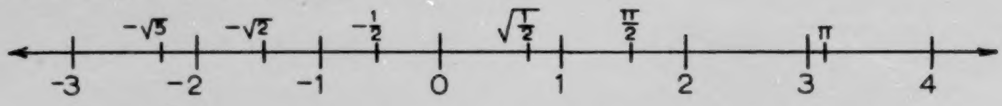
The numbers that we have so far are called the integers or whole numbers (positive, negative and zero). The counting numbers are the positive integers, and are often referred to by this name.

Of course, there are lots of points of the line that have no numbers attached to them so far. Our next step is to put in the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $-\frac{1}{2}$, $-\frac{1}{3}$, $-\frac{2}{3}$ and so on. The new numbers that we want to put in include all numbers that can be expressed as the ratio $\frac{p}{q}$ of any two integers (with q not equal to zero). We can indicate a few of these, as samples:



The numbers that we have so far are called the rational numbers. (This term is not supposed to mean that they are in a better state of mental health than other and less fortunate numbers. It merely refers to the fact that they are ratios of whole numbers).

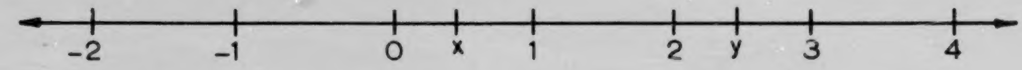
The rational numbers form a very large set. Between any two of them there is a third one; and there are infinitely many of them between any two whole numbers. It is a fact, however, that the rational numbers still do not fill up the line completely. For example, $\sqrt{2}$ is not rational; it cannot be expressed as the ratio of any two integers yet it does correspond to a point on the line. (For a proof, see Appendix III.) The same is true for $\sqrt{3}$ and $\sqrt{5}$, and also for such "peculiar" numbers as π. Such non-rational numbers are called irrational. If we insert all these extra numbers, in such a way that every point of the line has a number attached to it, then we have the real numbers. We indicate some samples, like this:



You should check that these numbers appear on the scale in approximately the positions where they belong. ($\sqrt{2}$ is approximately 1.41. How would you find $\sqrt{\frac{1}{2}}$?)

The real numbers will form part of the foundation of almost all that we are going to do in geometry. And it will be important throughout for us to think of the real numbers as being arranged on a line.

A number x is less than a number y if x lies to the left of y.



We abbreviate this by writing $x < y$. We notice that every negative number lies to the left of every positive number. Therefore, every negative number is less than every positive number. For example,

$$-1,000,000 < 1,$$

even though the number -1,000,000 may in a way look "bigger".

Expressions of the form $x < y$ are called inequalities. Any inequality can be written in reverse. For example,

$$1 > -1,000,000;$$

and in general, $y > x$ means that $x < y$.

The expression

$$x \leq y$$

means that x is less than or equal to y. For example, $3 \leq 5$ because $3 < 5$, and $5 \leq 5$, because $5 = 5$.

In your study of algebra, you have by now learned quite a lot about how the real numbers behave under addition and multiplication. All the algebra that you know can be derived from a few trivial-looking statements. These statements are the postulates for addition and multiplication of real numbers. You will find them listed in Appendix II. You may not have studied algebra on the basis of the postulates; and we are not going to start such a proceeding now. In this course, we are simply going to use the methods of elementary algebra, without comment.

We should be a little more careful, however, about inequalities and square roots. The relation $<$ defines an order for the real numbers. The fundamental properties of this order relation are the following:

O-1. (Uniqueness of Order) For every x and y, one and only one of the following conditions holds: $x < y$, $x = y$, $x > y$.

O-2. (Transitivity of Order) If $x < y$, and $y < z$, then $x < z$.

O-3. (Addition for Inequalities) If $x < y$, then $x + z < y + z$ for every z.

O-4. (Multiplication for Inequalities) If $x < y$ and $z > 0$, then $xz < yz$.

The statements O-2. and O-3. have an important consequence, which is worth mentioning separately:

O-5. If $a < b$ and $x < y$, then $a + x < b + y$.

This is true for the following reason: By O-3, we know that

$$a + x < b + x$$

and also that

$$b + x < b + y.$$

(That is, an inequality is preserved if we add the same number on each side.) By O-2, these last two inequalities fit together to give us

$$a + x < b + y,$$

which is what we wanted.

Finally, we are going to need the following property of the real numbers:

R-1. (Existence of Square Roots.) Every positive number has exactly one positive square root.

There is one rather tricky point in connection with square roots. When we say, in words, that x is a square root of a, this means merely that $x^2 = a$. For example, 3 is a square root of 9, and -3 is a square root of 9. But when we write, in symbols, that $x = \sqrt{a}$, we mean that x is the positive square root of a. Thus, the following statements are true or false, as indicated:

True: -3 is a square root of 9.

False: $-3 = \sqrt{9}$.

True: $\sqrt{9} = 3$.

False: $\sqrt{9} = \pm 3$.

The reason for this usage is simple, once you think of it. If $\sqrt{a}$ were allowed to denote either the positive root or the negative root, then we would have no way at all to write the positive square root of 7. (Putting a plus sign in front of the expression $\sqrt{7}$ gets us nowhere, because a plus sign never changes the value of an expression. If $\sqrt{7}$ were negative, then $+\sqrt{7}$ would also be negative).

Problem Set 2-2

1. Indicate whether each of the following is true or false.
   - a. The real number scale has no end points.
   - b. There exists a point on the real number scale which represents $\sqrt{2}$ exactly.
   - c. The point corresponding to $\frac{6}{7}$ on the real number scale lies between the points corresponding to $\frac{5}{6}$ and $\frac{7}{8}$.
   - d. Negative numbers are real numbers.
2. Restate the following in words:
   - a. $AB < CD$.
   - b. $x > y$.
   - c. $XY \geq YZ$.
   - d. $n \leq 3$.
   - e. $0 < 1 < 2$.
   - f. $5 \geq x \geq -5$.
   - g. $x > 0$.
3. Write as an inequality:
   - a. k is a positive number.
   - b. r is a negative number.
   - c. t is a number which is not positive.
   - d. s is a non-negative number.
   - e. g has a value between 2 and 3.
   - f. w has a value between 2 and 3 inclusive.
   - g. w has a value between a and b.
4. For which of the following is it true that $\sqrt{x^2} = x$?
   - a. $x = 5$.
   - b. $x = -5$.
   - c. $x = 0$.
   - d. $x = 7$.
   - e. $x = -1$.
   - f. $x > 0$.
   - g. $x < 0$.
   - h. $\frac{1}{x} > 0$.
5. How would the points corresponding to the following sets of numbers be arranged from left to right on a number scale in which the positive numbers are to the right of 0?
   - a. 3.1, 3.05, 3.009.
   - b. -2.5, -3, -1.5.
   - c. $\frac{5}{3}$, $1\frac{3}{5}$, $1\frac{5}{8}$.
   - d. $\frac{5}{3}$, $1\frac{3}{5}$, $-1\frac{5}{8}$.

*6. If r and s are real numbers, other than zero, and $r > s$, indicate whether the following are always true (T), sometimes true (S), or never true (N).

a. $s < r$.

b. $r - s > 0$.

c. $r - 2 < s - 2$.

d. $\frac{r}{s} > 1$.

e. $r^2 > s^2$.

*7. Follow the instructions of Problem 6 for the following:

a. $\frac{1}{r} > \frac{1}{s}$.

b. $\frac{1}{2} s < \frac{1}{2} r$.

c. $|r| > |s|$.

d. $r^3 > s^3$.

e. $1 - r < 1 - s$.

2-3. The Absolute Value

The idea of the absolute value of a number is easily understood from a few examples:

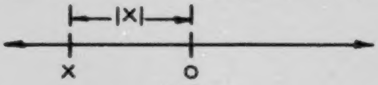
(1) The absolute value of 5 is 5.

(2) The absolute value of -5 is 5.

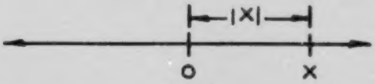
(3) The absolute value of π is π.

(4) The absolute value of $-\pi$ is π, and so on.

Graphically speaking, the absolute value of x is simply the distance between 0 and x on the number scale, regardless of whether x lies to the left or to the right of 0. The absolute value of x is written as $|x|$.



$x < 0$



$x > 0$

The two possibilities for x are indicated in the figures. In each of the two cases, $|x|$ is the distance between 0 and x.

If a particular number is written down arithmetically, it is easy to see how we should write its absolute value. The reason is that in arithmetic, the positive numbers are written as 1, 2, 3, 4, and so on. A way to write negative numbers is to put minus signs in front of the positive numbers. This gives -1, -2, -3, -4, and so on. Therefore, in arithmetic, if we want to write the absolute value of a negative number, we merely omit the minus sign, thus, $|-1| = 1$ $|-2| = 2$, and so on.

We would like to give an algebraic definition for $|x|$, and we would like the definition to apply both when x is positive and when x is negative. In algebra, of course, the letter x can represent a negative number. In working algebra problems, you have probably written $x = -2$ nearly as often as you have written $x = 2$. If x is negative, then we can't write the corresponding positive number by omitting the minus sign, because there isn't any minus sign to omit. There is a simple device, however, that gets around our difficulty: if x is negative, then the corresponding positive number is $-x$. Here are some examples:

$x = -1$, $-x = -(-1) = 1$; that is, if $x = -1$, then $-x = 1$.

$x = -2$, $-x = -(-2) = 2$; that is, if $x = -2$, then $-x = 2$.

$x = -5$, $-x = -(-5) = 5$; that is, if $x = -5$, then $-x = 5$.

In each of these cases, x is negative and $-x$ is the corresponding positive number. And in fact, this is what always happens. Since we knew all along that $|x| = x$ when x is positive or zero, it follows that the absolute value is described by the following two statements:

(1) If x is positive or zero, then $|x| = x$.

(2) If x is negative, then $|x| = -x$.

If this still looks doubtful to you, try substituting various numbers for x. No matter what number x you pick, one of the conditions above will apply, and will give you the right answer for the absolute value.

Problem Set 2-3

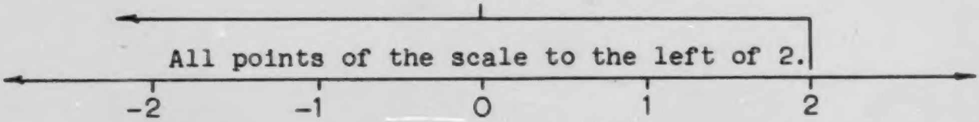
1. Indicate which of the following are always true:
   - a. $|-3| = 3$.
   - b. $|3| = -3$.
   - c. $|2 - 7| = |7 - 2|$.
   - d. $|0 - 5| = |5 - 0|$.
   - e. $|n| = n$.

*2. Indicate which of the following are always true:
   - a. $|-n| = n$.
   - b. $|n^2| = n^2$.
   - c. $|y - x|^2 = y^2 - 2xy + x^2$.
   - d. $|a - 2| = |2 - a|$.
   - e. $|d| + 1 = |d + 1|$.

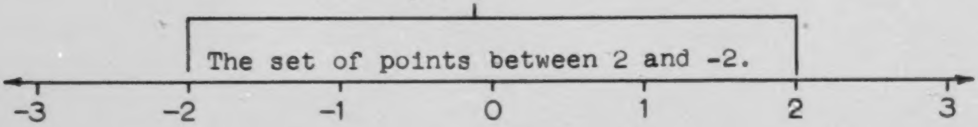
3. Complete these statements:
   - a. If $0 < r$, then $|r| =$ ___.
   - b. If $0 > r$, then $|r| =$ ___.
   - c. If $0 = r$, then $|r| =$ ___.

4. The following three examples give a geometric interpretation to algebraic statements.

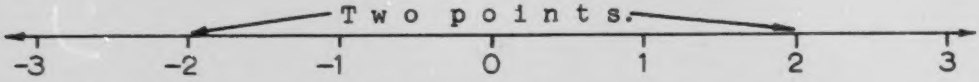
$x < 2$.


All points of the scale to the left of 2.
-2
-1
0
1
2

$|x| < 2$.


The set of points between 2 and -2.
-3
-2
-1
0
1
2
3

$|x| = 2$.


Two points.
-3
-2
-1
0
1
2
3

Continue as above for the following problems:

a. $x < 0$.

b. $x = 1$.

c. $x > 1$.

d. $x \leq 1$.

e. $|x| = 1$.

f. $|x| \leq 1$.

g. $|x| > 1$.

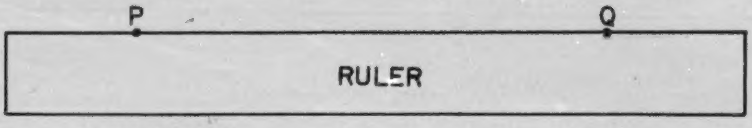
h. $|x| \geq 0$.

5. a. How would the set of points represented by $x \geq 0$ differ from the set represented by $x > 0$?

b. How would the set of points represented by $0 \leq x \leq 1$ differ from the set represented by $0 < x < 1$?

2-4. *Measurement of Distance*

The first step in measuring the distance between two points P and Q is to lay down a ruler between them, like this:



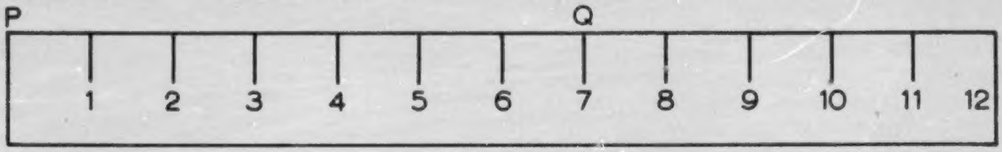
Of course we want to use a *straight* ruler, since we cannot expect to get consistent results if our rulers are curved or notched. A straight ruler has the property that however it is placed between P and Q the line drawn along its edge is always the same. In other words, this line is completely determined by the two given points. We express this basic property of lines as our first geometric postulate:

> *Postulate 1*. Given any two different points, there is exactly one line which contains both of them.

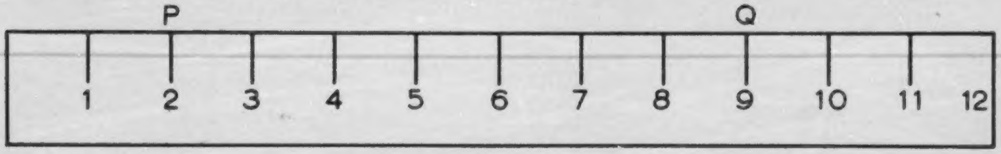
We shall often refer to this postulate, briefly, by saying that every two points *determine* a line. This is simply an abbreviated way of stating *Postulate 1*.

To designate the line determined by two points P and Q we use the notation $\overleftrightarrow{PQ}$. (The double arrow will recall our picture of the line.) Of course we can always abbreviate by introducing a new letter and calling the line L, or W, or anything else.

Now let us consider the marks on the ruler and the actual distance between P and Q. The easiest way to measure the distance is to place the ruler like this:

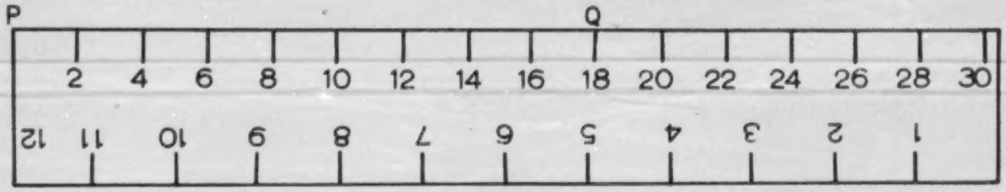


This gives 7". Of course, there is no need to put one end of the ruler at P. We might put it like this:



In this case, the distance between P and Q, measured in inches, is 9 - 2 = 7, as before.

On many rulers that are sold now, one edge is laid off in inches, and the other edge in centimeters. Using the centimeter scale, we can measure the distance between P and Q like this:



This gives the distance as approximately 18 cm., where cm. stands for centimeters.

A foot is, of course, 12", and a yard is 36". A meter is a hundred centimeters; m. stands for meters. A millimeter is a tenth of a centimeter (or $\frac{1}{1000}$ of a meter); mm. stands for millimeters. We can therefore measure the distance between

P and Q in at least this many ways: 18 cm., 180 mm., .18 m., 7 in., $\frac{7}{12}$ ft., $\frac{7}{36}$ yds.

That is, the number we get, as a measure of the distance, depends on the unit of measure. We can use any unit we like, as long as we use it consistently, and as long as we say what unit we are using.

Problem Set 2-4

1. What common fractions (or integers) are needed to complete the following table?

   a. 2 in. = ___ft. = ___yd.

   b. ___ in. = $4\frac{1}{2}$ ft. = ___yd.

   c. ___ in. = ___ft. = $\frac{2}{3}$ yd.

2. What numbers are needed to complete the following table?

   a. 500 mm. = ___cm. = ___m.

   b. ___ mm. = 32.5 cm. = ___m.

   c. ___ mm. = ___cm. = 7.32 m.

3. a. Suppose you decide to use the width of an $8\frac{1}{2}$ in. by 11 in. sheet of paper as a unit of length. What is the length and the width of the sheet in terms of this unit?

   b. Repeat the problem with the length of the sheet as your new unit.

4. If the lengths of the sides of a triangle are 3 ft., 4 ft., and 5 ft., it is a right triangle because $3^2 + 4^2 = 5^2$. Verify that the Pythagorean relationship still holds if the lengths above are expressed in inches.

5. If the length of each side of a square is 4 ft. its perimeter is 16 ft. and its area is 16 sq. ft. Observe that the numerical value of the perimeter is equal to the numerical value of the area.

   a. Show that the numerical values of the perimeter and area will no longer be equal to each other if the length of the side is expressed in inches.

b. In yards.

*6. Generalize Problem 4. Given that the numbers a, b and c are the number of units in the sides of a triangle if some particular unit of length is used and that $a^2 + b^2 = c^2$. Show that the Pythagorean relationship will still hold if the unit of length is multiplied by n. (Hint: The lengths of the sides will become $\frac{a}{n}$, $\frac{b}{n}$ and $\frac{c}{n}$. If a, b and c seem too abstract use 3, 4 and 5 at first.)

*7. Generalize Problem 5. Show that if the numerical values of the area and perimeter of a square are equal for some particular unit of measure, then they will not be equal for any other unit. (Hint: Start by letting the number s be the length of the side of the square for some unit and equating the area and perimeter formulas.)

2-5. A Choice of a Unit of Distance

We have noticed that the choice of a unit of distance is merely a matter of convenience. Logically speaking, one unit works as well as another, for measuring distances. Let us therefore choose a unit, and agree to talk in terms of this unit in all of our theorems. (It will do no harm to think of our unit as being anything we like. If you happen to like inches, feet, yards, centimeters, cubits, or furlongs, you are free to consider that these are the units that we are using. All of our theorems will hold true for any unit.)

Thus, to every pair of points, P, Q there will correspond a number which is the measure of the distance between P and Q in terms of our unit. Such numbers will be used extensively in our work, and it would be very inconvenient to have to be continually repeating the long phrase "measure of the distance between P and Q in terms of our unit". We shall therefore shorten this phrase to "distance between P and Q", trusting that you will be able to fill in the remaining words if it should ever be necessary.

We can now describe this situation in the following precise form:

Postulate 2. (The Distance Postulate.) To every pair of different points there corresponds a unique positive number.

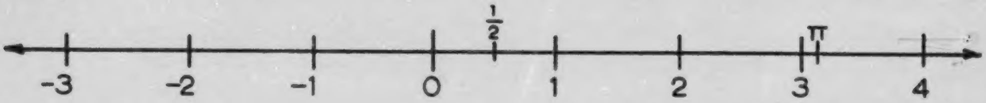
Definition. The distance between two points is the positive number given by the Distance Postulate. If the points are P and Q, then the distance is denoted by PQ.

It will sometimes be convenient to allow the possibility P = Q, that is, P and Q are the same point; in this case, of course, the distance is equal to zero. Notice that distance is defined simply for a pair of points, and does not depend on the order in which the points are mentioned. Therefore PQ is always the same as QP.

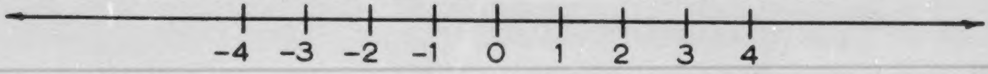
Some of the problems you will be asked to do will involve various units of distance, such as feet, miles, meters, etc. As noted above, our theorems will be applicable to any of these units, provided you consistently use just one unit throughout any one theorem. You can use inches in one theorem and feet in another, if you wish, but not both in the same theorem.

2-6. An Infinite Ruler

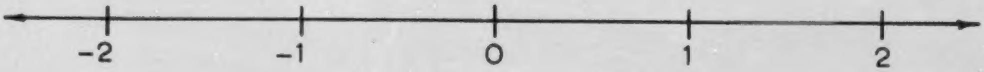
At the beginning of this chapter we laid off a number-scale on a line, like this:



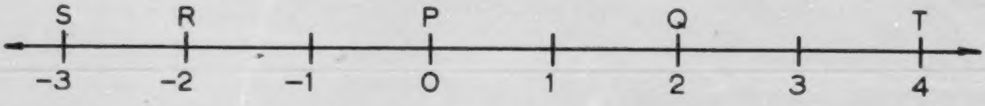
We could, of course, have compressed the scale, like this:



or stretched it, like this:



But let us agree, from now on, that every number-scale that we lay off on a line is to be chosen in such a way that the point labeled x lies at a distance |x| from the point labeled 0. For example, consider the points P, Q, R, S, and T, labeled with the numbers 0, 2, -2, -3, and 4, as in the figure below:



Then PQ = 2, PR = 2, PS = 3 and PT = 4.

If we examine various pairs of points on the number-scale, it seems reasonable to find the distance between two points by taking the difference of the corresponding numbers. For example,

$$PQ = 2, \quad \text{and } 2 = 2 - 0;$$
$$QT = 2, \quad \text{and } 2 = 4 - 2;$$
$$SQ = 5, \quad \text{and } 5 = 2 - (-3);$$
$$RT = 6, \quad \text{and } 6 = 4 - (-2).$$

Notice, however, that if we look at the pairs of points in reverse order, and perform the subtractions in reverse order, we will get the wrong answer every time: instead of getting the distance (which is always positive), we will get the corresponding negative number. This difficulty, however, is easy to get around. All we need to do is to take the absolute value of the difference of the numbers. If we do this, then all of our positive right answers will still be right, and all of our negative wrong answers will become right.

Thus we see that the distance between two points is the absolute value of the difference of the corresponding numbers.

Surely all this seems reasonable. But surely we have not proved it on the basis of the only postulatesthat we have written down so far. (And, in fact, it cannot be proved on the basis of the Distance Postulate.) We therefore sum up the above discussion in the form of a new postulate, like this:

Postulate 3. (The Ruler Postulate.) The points of a line can be placed in correspondence with the real numbers in such a way that

(1) To every point of the line there corresponds exactly one real number,

(2) To every real number there corresponds exactly one point of the line, and

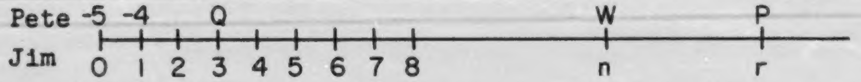
(3) The distance between two points is the absolute value of the difference of the corresponding numbers.

We call this the Ruler Postulate because in effect it furnishes us with an infinite ruler, with a number-scale on it, with which we can measure distances on any line.

Definitions. A correspondence of the sort described in Postulate 3 is called a coordinate system for the line. The number corresponding to a given point is called the coordinate of the point.

Problem Set 2-6

1. Simplify:
   a. $|3 - 6|$.
   b. $|6 - 3|$.
   c. $|-2 - 1|$.
   d. $|-4 - (-2)|$.
   e. $|a - (-a)|$.
   f. $|a| - |-a|$.
2. Using the kind of coordinate system discussed in the text, find the distance between point pairs with the following coordinates:
   a. 0 and 12.
   b. 12 and 0.
   c. 0 and -12.
   d. -12 and 0.
   e. $-3\frac{1}{2}$ and -5.
   f. -5.1 and 5.1.
   g. $\sqrt{2}$ and $\sqrt{3}$.
   h. x_1 and x_2.
   i. 2a and -2a.
   j. r - s and r + s.
3.



   The lower numbering on this scale was put there by Jim. Pete began the upper numbering but quit.
   a. Copy the scale and write in the rest of Pete's numbering.
   b. Show how to find the distance from P to Q, first by using Jim's scale and then by using Pete's scale.
   c. Do the same for the distance from W to P.
4. Suppose in measuring the distance between two points P and Q you intended to place the zero of the number-scale at P and read a positive value at Q. However, you happen to place the number-scale so that P is at $\frac{1}{4}$ and Q is farther to the right.

How is it still possible to measure the distance PQ?

*5. Consider a coordinate system of a line. Suppose 2 is added to the coordinate of each point and this new sum is assigned to the point.

a. Will each point then correspond to a number and each number to a point?

b. If two points of the line had coordinates p and q in the coordinate system given, what numbers are assigned to them in the new numbering?

c. Show that the formula

| (Number assigned to one point)-(Number assigned to other point) | gives the distance between the two points.

d. Does the new correspondence between points and numbers satisfy each of the three conditions of Postulate 3? (If it does it may be called a coordinate system.)

*6. Suppose a coordinate system is set up on a line so that each point P corresponds to a real number n. If we replace each n by $-n$, then the point P will correspond to a number $-n$. Show that this correspondence is also a coordinate system for the line. (HINT: It is apparent that each point will have a number associated with it and each number a point. You must show in addition that the absolute value of the difference of the numbers assigned to the two points will remain unchanged when the numbering is changed.)

7. In a certain county the towns of Alpha, Beta and Gamma are collinear (on a line) but not necessarily in that order. It is 16 miles from Alpha to Beta and 25 miles from Beta to Gamma.

a. Is it possible to tell which town is between the other two? Which town is not between the other two?

b. There might be two different values for the distance from Alpha to Gamma. Use a sketch to determine what these are.

c. If you are given the additional information that the distance from Alpha to Gamma is 9 miles, then which town is between the other two?

d. If the distance between Alpha and Beta were r miles, the distance from Alpha to Gamma s miles, and the distance from Beta to Gamma r + s miles, which city would be between the other two?

8. A, B, C are three collinear points. A and B are 10" apart, and C is 15" from B. Is there just one way to arrange these points? Explain.

9. Three different coordinate systems are assigned to the same line. Three fixed points A, B, C of the line are assigned values as follows:

   With the first system the coordinate of A is -6 and that of B is -2.

   With the second the coordinates of A and C are 4 and -3 respectively.

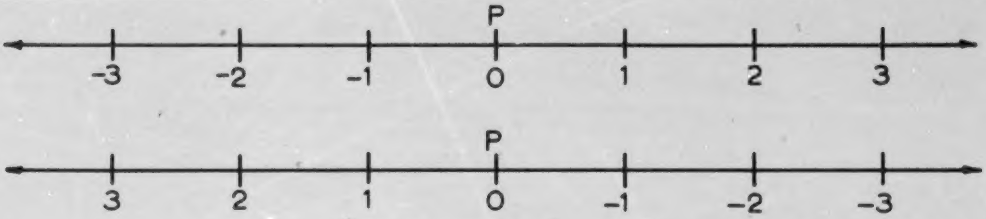
   With the third the respective coordinates of C and B are 7 and 4.

   What point is between the other two?

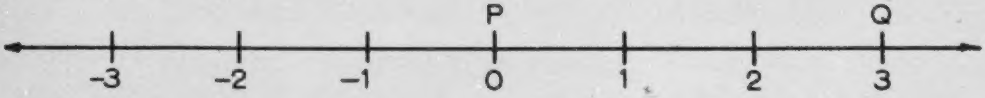
   Evaluate AB + BC + AC.

2-7. The Ruler Placement Postulate - Betweenness - Segments And Rays

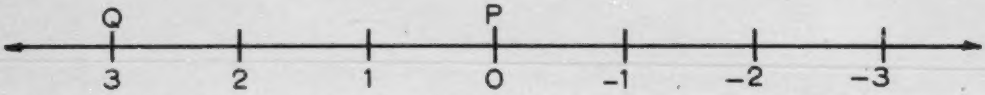
The Ruler Postulate (Postulate 3) tells us that on any line, we can set up a coordinate system by laying off a number-scale. This can be done in lots of different ways. For example, given a point P of the line, we can start by making P the zero-point. And we can then lay off the scale in either direction, like this:



This means that given another point Q of the line, we can always choose the coordinate system in such a way that Q corresponds to a positive number, like this:



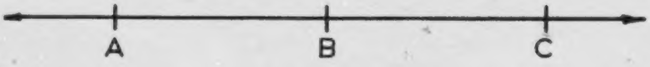
or this:



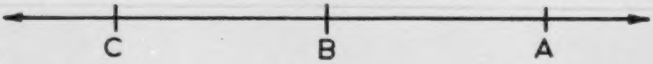
Let us write this down, for future reference, in the form of a postulate.

Postulate 4. (The Ruler Placement Postulate.) Given two points P and Q of a line, the coordinate system can be chosen in such a way that the coordinate of P is zero and the coordinate of Q is positive.

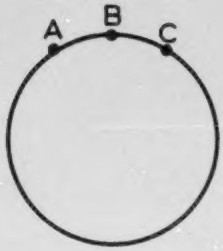
Everybody knows what it means to say that a point B is between two points A and C. It means that A, B and C lie on the same line, and that they are arranged on the line like this:

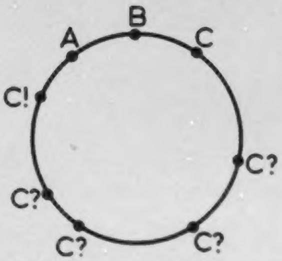


or like this:



If we are going to use betweenness as a mathematical idea, however, we had better give a mathematical definition that states exactly what we mean, because the feelings that we have in our bones are not necessarily reliable. To see this, let us look at the corresponding situation on a circle. In the figure on the left,





it seems reasonable to say that B is between A and C. But C can be moved around the circle in easy stages, without passing over A or B, so as to lie just to the left of A, as in the right-hand figure. In the final position, indicated by the exclamation point, it looks as if A is between B and C. In this respect, circles are tricky. Given any three points of a circle, it is quite reasonable to consider that each of them is between the other two.

Betweenness on a line is not at all tricky. It is easy to say exactly what it means for one point of a line to be between two others. We can do this in the following way:

Definition. B is between A and C if (1) A, B and C are distinct points on the same line and (2) $AB + BC = AC$.

It is easy to check that this definition really expresses our common-sense idea of what betweenness ought to mean. It may be a good idea to explain, however, the way in which language is ordinarily used in mathematical definitions. In the definition of betweenness, two statements are connected by the word if. What we really mean is that the statements before and after the word if are completely equivalent. Whenever, in some theorem or problem, we are given or can prove that conditions (1) and (2) both hold, then we can conclude that B is between A and C. And whenever we find that B is between A and C then we can conclude that (1) and (2) both hold. This is not a strictly logical use of the word if, and in particular the word if is never used in this way in postulates, theorems or problems. In definitions, however, it is common.

The following theorem describes betweenness in terms of coordinates on a line.

Theorem 2-1. Let A, B, C be three points of a line, with coordinates x, y, z. If

$$x < y < z,$$

then B is between A and C.

Proof: Since $x < y < z$, we know that the numbers $y - x$, $z - y$, and $z - x$ are all positive. Therefore, by definition of the absolute value,

$$\begin{aligned} |y - x| &= y - x, \\ |z - y| &= z - y, \\ |z - x| &= z - x. \end{aligned}$$

Therefore, by the Ruler Postulate,

$$\begin{aligned} AB &= y - x, \\ BC &= z - y, \\ AC &= z - x. \end{aligned}$$

Therefore

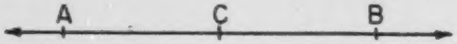
$$\begin{aligned} AB + BC &= (y - x) + (z - y) \\ &= -x + z \\ &= z - x \\ &= AC. \end{aligned}$$

Therefore, by the definition of betweenness, B is between A and C, which was to be proved.

Problem Set 2-7a

1. a. A number-scale is placed on a line with -3 falling at R and 4 at S. If the Ruler Placement Postulate is applied with 0 placed on R and a positive number on S, what will this number be?
   b. Same question if -4 falls at R and -10 at S.
   c. Same question if 8 falls at R and -2 at S.
   d. Same question if $-4\frac{1}{2}$ falls at R and 4 at S.
   e. Same question if 5.2 falls at R and 6.1 at S.
   f. Same question if x_1 falls at R and x_2 at S.
2. Explain briefly how the Ruler Placement Postulate simplifies the procedure given by the Ruler Postulate for computation of distance between two points.

3. Suppose R, S and T are collinear points. What must be true of the lengths RS, ST and RT if S is to be between R and T? (See definition of between.)

4. 

AC and BC each equals 8. The coordinate of C is 6. The coordinate of B is greater than the coordinate of C. What are the coordinates of A and B?

5. If a, b and c are coordinates of collinear points, and if $|a - c| + |c - b| = |a - b|$, what is the coordinate of the point which lies between the other two? Be able to justify your answer.

6. If x_1, x_2 and x_3 are coordinates of points on a line such that $x_3 > x_1$ and $x_2 < x_1$, which point is between the other two? Which theorem would be used to prove your answer?

7. Consider a coordinate system in which A is assigned the number 0, B is assigned the positive number r, E the number $\frac{1}{3}r$, and F the number $\frac{2}{3}r$.
Prove that:
a. AE = EF = FB
b. E is between A and F.

*8. Prove: If A, B and C are three points of a line with coordinates x, y and z respectively and if $x > y > z$, then B is between A and C.

Theorem 2-2. Of any three different points on the same line, one is between the other two.

Proof: Let the points be A, B and C. By the Ruler Postulate, there is a coordinate system for the line. Let the coordinates of A, B, and C be x, y, and z. There are now six possibilities:

(1) $x < y < z$,
(2) $x < z < y$,
(3) $y < x < z$,
(4) $y < z < x$,
(5) $z < x < y$,
(6) $z < y < x$.

In each of these cases, Theorem 2-2 follows by Theorem 2-1. In cases (1) and (6), B is between A and C. In cases (2) and (4), C is between A and B. In cases (3) and (5), A is between B and C.

Theorem 2-3. Of three different points on the same line, only one is between the other two.

Restatement. If A, B and C are three different points on the same line, and B is between A and C, then A is not between B and C, and C is not between A and B.

(It often happens that a theorem is easier to read, and easier to refer to, if it is stated in words. But to prove theorems, we usually need to set up a notation, giving names to the objects that we will be talking about. For this reason, we shall often give restatements of theorems, in the style that we have just used for Theorem 2-3. The restatement gives us a sort of head-start in the proof.)

Proof: If B is between A and C, then

$$AB + BC = AC.$$

If A is between B and C, then

$$BA + AC = BC.$$

What we need to prove is that these two equations cannot both hold at the same time.

If the first equation holds, then

$$AC - BC = AB.$$

If the second equation holds, then

$$AC - BC = -BA = -AB.$$

Now AB is positive, and -AB is negative. Therefore, these equations cannot both be true, because the number AC - BC cannot be both positive and negative.

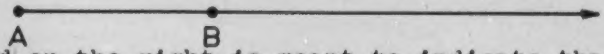
In an entirely similar manner we can show that C is not between A and B.

Definitions. For any two points A and B the *segment* $\overline{AB}$ is the set whose points are A and B, together with all points that are between A and B. The points A and B are called the *end-points* of $\overline{AB}$.

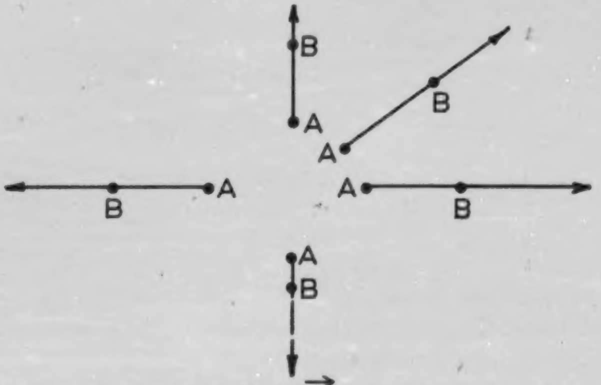
Notice that there is a big difference between the segment $\overline{AB}$ and the distance AB. The segment is a geometrical figure, that is, a set of points. The distance is a number, which tells us how far A is from B.

Definition. The distance AB is called the *length* of the segment $\overline{AB}$.

A ray is a figure that looks like this:



The arrow-head on the right is meant to indicate that the ray includes *all* points on the line to the right of the point A, plus the point A itself. The ray is denoted by $\overrightarrow{AB}$. Notice that when we write $\overrightarrow{AB}$, we simply mean the ray that starts at A, goes through B, and then goes on in the same direction forever. The ray might look like any of the following:

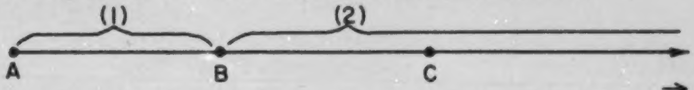


That is, the arrow in the symbol $\overrightarrow{AB}$ always goes from left to right, regardless of how the ray is pointed in space.

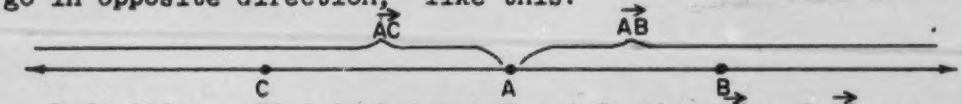
Having explained informally what we are driving at, we proceed to give an exact definition.

Definitions. Let A and B be points of a line L. The ray $\overrightarrow{AB}$ is the set which is the union of (1) the segment $\overline{AB}$ and (2) the set of all points C for which it is true that B is between A and C. The point A is called the end-point of $\overrightarrow{AB}$.

These two parts of the ray are as indicated:



If A is between B and C on L, then the two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ "go in opposite direction," like this:



Definition. If A is between B and C, then $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are called opposite rays.

Note that a pair of points A, B determines six geometric figures:

The line $\overleftrightarrow{AB}$,

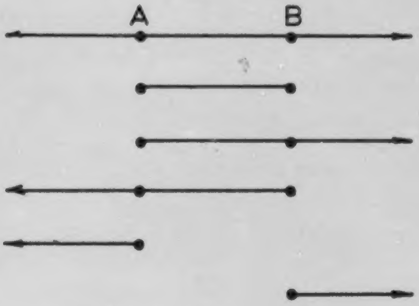
The segment $\overline{AB}$,

The ray $\overrightarrow{AB}$,

The ray $\overrightarrow{BA}$,

The ray opposite to $\overrightarrow{AB}$,

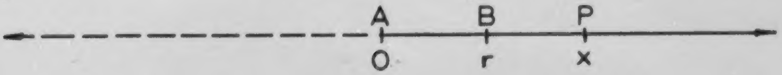
The ray opposite to $\overrightarrow{BA}$.



The Ruler Placement Postulate has three more simple and useful consequences.

Theorem 2-4. (The Point Plotting Theorem) Let $\overrightarrow{AB}$ be a ray, and let x be a positive number. Then there is exactly one point P of $\overrightarrow{AB}$ such that AP = x.

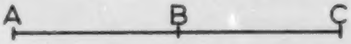
Proof: By the Ruler Placement Postulate, we can choose the coordinate system on the line $\overleftrightarrow{AB}$ in such a way that the coordinate of A is equal to 0 and the coordinate of B is a positive number r:



(In the figure, the labels above the line represent points, and the labels below the line represent the corresponding numbers.)

Let P be the point whose coordinate is x. Then P belongs to $\overrightarrow{AB}$, and $AP = |x - 0| = |x| = x$, because x is positive. Since only one point of the ray has coordinate equal to x, only one point of the ray lies at a distance x from A.

Definition. A point B is called a mid-point of a segment $\overline{AC}$ if B is between A and C, and $AB = BC$.



Theorem 2-5. Every segment has exactly one mid-point.

Proof. On the segment $\overline{AC}$ we want a point B such that $AB = BC$. We know, by definition of a segment, that B is between A and C. Therefore, $AB + BC = AC$. From these two equations we conclude that $2AB = AC$, or $AB = \frac{1}{2} AC$. Since B is to lie on segment $\overline{AC}$ it must also lie on ray $\overrightarrow{AC}$, and Theorem 2-4 tells us that there is exactly one such point B.

Definition. The mid-point of a segment is said to bisect the segment. More generally, any figure whose intersection with a segment is the mid-point of the segment is said to bisect the segment.

Problem Set 2-7b

1. If three points are on a line, how many of them are not between the other two?
2. Each of the following is a particular case of what definition or theorem?

   If three collinear points R, S and T have coordinates respectively 4, 5 and 8:

   a. S is between R and T because $4 < 5$ and $5 < 8$.
   b. R cannot be between S and T since S is between R and T.
   c. S is between R and T because $RS + ST = RT$.

3. Describe in mathematical language what points are included in:
   a. $\overline{XY}$ b. $\overrightarrow{XY}$

*4. Show that the restriction "between A and C" in the definition of the midpoint of $\overline{AC}$ is unnecessary by proving the following theorem:

   If B is any point on the line $\overleftrightarrow{AC}$ such that $AB = BC$, then B is between A and C. (Hint: Show that A cannot be between B and C nor C between A and B. Use algebra in showing this. Use Theorem 2-2 to finish the proof.)

*5. Suppose that P is a point on a line M and r is a positive number. Which of the previous theorems shows that there are exactly two points on M whose distance from P is the given number r?

*6. Prove that if B is between A and C, then $AC > AB$.

7. a. Copy the following paragraph. Supply the appropriate missing symbol, if any, over each letter pair.

   XZ contains points Y and R, but XZ contains neither points Y nor R. R belongs to XZ but Y does not.
   YZ + ZR = YR.

   b. Make a drawing showing the relative position of the four points.

Review Problems

1. Consider the following sets:

   S_1 is the set of all boys in the 10th grade.
   S_2 is the set of all girls in the 10th grade.
   S_3 is the set of all 10th grade geometry students.
   S_4 is the set of all students in high school.
   S_5 is the set of all 10th grade students.

   a. What is the intersection of S_1 and S_5?
   b. What is the union of S_3 and S_4?
   c. What is the intersection of S_3 and S_4?

d. What is the union of S_1 and S_2?

e. What is the intersection of S_1 and S_2?

2. a. How many squares does a given positive number have?

b. How many square roots?

c. Is $\sqrt{3}$ ever negative?

3. a. Draw a line and locate the following points on it. (The coordinate of each point is given in parentheses.) Use any unit of measure you choose, but use the same unit throughout.

P (2), Q (-1), R (0), S (-3), T (4).

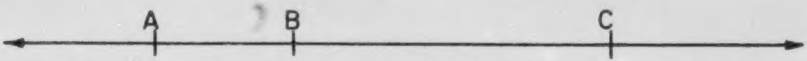
b. Find PQ, RT, TR, PT, QS.

4. a. If $a > b$, then $a - b$ is __________.

b. If $0 < k$ and $k^2 < 4$, then k is __________.

c. If $a < b$ then $a - b$ is _________.

5.



a. Write an equation that describes the relative positions of these three points.

b. Under what condition would B be the midpoint of $\overline{AC}$?

6. Four points A, B, C, D are arranged along a line so that $AC > AB$ and $BD < BC$. Picture the line with the four points in place. Is there more than one possible order? Explain.

7. The letter pairs contained in the following paragraph are either numbers, lines, line segments, or rays. Indicate which each is by placing the proper missing symbol, if any, above each letter pair.

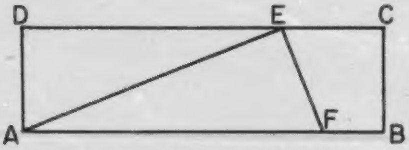
"AB + BC = AC. DB contains points A and C, but DB contains neither point A nor point C. A belongs to DB but C does not." Draw a picture that illustrates your response.

8. A is the set of all integers x and y whose sum is 13. B is the set of all integers whose difference is 5. What is the intersection of A and B?

9. John said, "My house is on West Street halfway between Bill's house and Joe's house." Pete said, "So is mine!" What can you conclude concerning John and Pete?

10. N men sit on a straight bench. Of how many may it be said, "He sits between two people?"

11. Use the figure below to answer questions a. through e.:



a. Describe the intersection of triangle AEF and rectangle ABCD.

b. Describe the intersection of segment $\overline{EF}$ and rectangle ABCD.

c. Describe the union of segments $\overline{AF}$, $\overline{EF}$, and $\overline{AE}$.

d. Describe the intersection of segments $\overline{AE}$ and $\overline{BC}$.

e. Describe the union of triangle AEF and segment $\overline{AE}$.

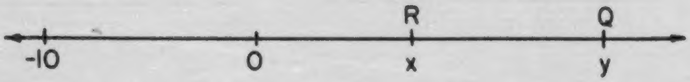
12. Given a group of five men (Messrs. Andrews, Brown, Crawford, Douglas, and Evans). a. From the group, how many different 4-man committees can be formed? b. 2-man? c. 3-man?

13. Given that A, B and C are collinear and that $AB = 3$ and $BC = 10$, can $AC = 6$? Give a picture to explain your answer.

14. Indicate which of the following statements are true and which are false. For any that are false, give a correct answer.

a. $|-13+7| = 20$.
b. $|-8-9| = 17$.
c. $|5a-6a| = |a|$.
d. $|9+2| = 11$.
e. $|-4| - |-11| = -7$.
f. $|(3a-6) - (a-7)| = |2a+1|$.
g. $|7| - |9| = -2$.
h. $|-11| - |-4| = -7$.

15.



Looking at this number-scale, Jack said, "The length of $\overline{RQ}$ is $|x-y|$." Sam maintained that when giving the length of $\overline{RQ}$ it would be just as correct to use simply $y-x$.

Do you agree with Sam? Explain.

16. The first numbering of the points on the line below represents a coordinate system. Which of the other numberings are not coordinate systems according to Postulates 2 and 3?

	-3	-2	-1	0	1	2	3	4	5	6	7
a.	-7	-6	-5	-4	-3	-2	-1	0	1	2	3
b.	0	1	2	3	4	5	4	3	2	1	0
c.	11	12	13	14	15	16	17	18	19	20	21
d.	-11	-12	-13	-14	-15	-16	-17	-18	-19	-20	-21
e.	-3	-2	1	0	-1	2	3	4	5	6	7

*17. Consider the points of a line whose coordinates are described as follows:

a. $x < 3$.
b. $x = 1$.
c. $x > 2$.
d. $x \leq 1$.
e. $x = -3$.
f. $|x| \leq 3$.
g. $|x| > 2$.
h. $|x| \geq 0$.

Which of the above sets is a ray? A point? A line? A segment?

Chapter 3

LINES, PLANES AND SEPARATION

3-1. Lines and Planes in Space.

In the last chapter, we were talking only about lines and the measurement of distance. We shall now proceed to the study of planes and space. We recall that our basic undefined terms are point, line and plane. Every line is a set of points, and every plane is a set of points.

Definition. The set of all points is called space.

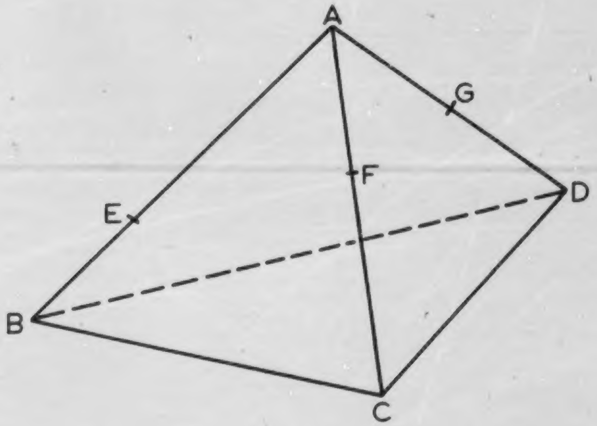
In this section we will explain some of the terms we are going to use in talking about points, lines and planes, and state some of the basic facts about them. Most of these basic facts will be stated as postulates. Some of them will be stated as theorems. These theorems will be so simple that it would be reasonable to accept them without proof, and call them postulates. We do not do this, however; the first of them is going to be proved in this section, and the rest of them will be proved, on the basis of the postulates, in a later chapter. For the present, however, let us not worry about this question, one way or another; let us simply try to get these basic facts straight.

Problem Set 3-1a

1. On a piece of paper, or on the blackboard, place two marks to represent points A and B. How many different lines can you draw through both A and B? What happens if you consider "line" in a sense other than "straight"?
2. Take a piece of stiff cardboard or your book. Can you support it in a fixed position on the ends of two pencils? What is the minimum number of pencils needed to support it in this way?
3. Think of one cover of your book as part of a plane. How many points are needed to determine this plane?
4. How many end-points does a line have? How many end-points does a line segment have?

Definition. A set of points is collinear if there is a line which contains all the points of the set.

Definition. A set of points is coplanar if there is a plane which contains all the points of the set.



For example, in the above figure of a triangular pyramid, A, E and B are collinear, and A, F and C are collinear, but A, B and C are not collinear. A, B, C and E are coplanar, and A, C, D, F and G are coplanar, but A, B, C and D are not coplanar.

One of the properties we desire for the sets of points which we call lines, planes and space is that they should contain lots of points. Also, a plane should in some sense be "bigger" than a line and space should be "bigger" than any plane. The existence of plenty of points on a line is insured by the Ruler Postulate; for planes and space the following postulate will give us the properties we want:

Postulate 5. (a) Every plane contains at least three non-collinear points.

(b) Space contains at least four non-coplanar points.

For convenience in reference we repeat Postulate 1.

Postulate 1. Given any two different points, there is exactly one line which contains them.

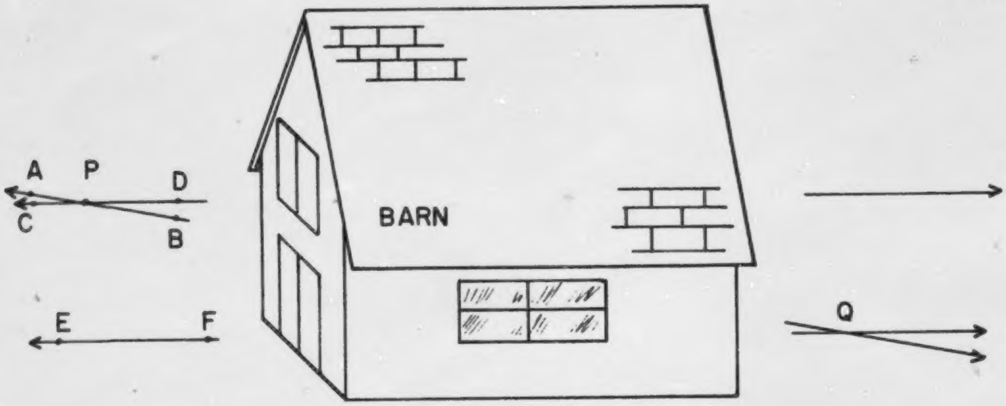
Theorem 3-1. Two different lines intersect in at most one point.

The proof of this follows from Postulate 1. It is impossible for two different lines to intersect in two different points P and Q because by Postulate 1 there is only one line that contains P and Q.

Problem Set 3-1b

1. Given: 1. L_1 and L_2 are different lines.
   2. Point P lies on L_1 and L_2.
   3. Point Q lies on L_1 and L_2.

   What can you say must be true about P and Q?
2. How many lines can contain one given point? two given points? any three given points?
3. The diagram shows three different lines $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$, and $\overleftrightarrow{EF}$, whose view is partially obstructed by a barn. If $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect to the left of the barn, which postulate says that they cannot also intersect to the right of the barn?



4. Draw a diagram to illustrate each part of this problem and justify your answers in terms of Postulate 1.
   a. How many lines can be drawn through both of two fixed points?

b. How many lines can be drawn through three points taken two at a time?

5. a. How many lines can be drawn through four coplanar points, taken two at a time, if no three of the points are collinear? (Hint: Call the points A, B, C, D.)

   b. How many lines would there be if points A, B, and C were collinear?

   c. Draw a diagram for (a) and (b).

*6. "A point lies on a line" and "a line contains a point" are two forms of saying the same thing.

   a. The definitions of collinear and coplanar are phrased using the second form. Rephrase these definitions using the first form.

   b. The first part of Postulate 5 is phrased using the second form. Rephrase this part of Postulate 5 using the first form.

*7. As in Problem 6, Postulate 1 is written in one of the two forms. Which form? Restate Postulate 1 in the other form.

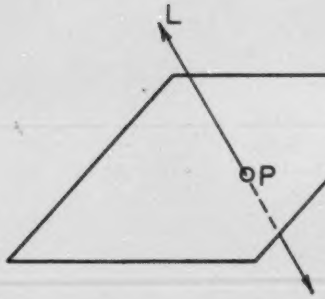
By Postulate 5 a plane contains at least three points. Does it contain any more? On the basis of our present postulates we cannot conclude that it does, so we introduce

> Postulate 6. If two points lie in a plane, then the line containing these points lies in the same plane.

This postulate essentially says that a plane is flat, that is, that if it contains part of a line it contains the whole line.

Theorem 3-2. If a line intersects a plane not containing it, then the intersection is a single point.

This follows from Postulate 6 in the same way that Theorem 3-1 follows from Postulate 1.

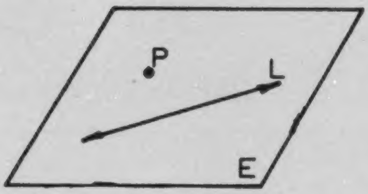


The figure shows line L, intersecting a plane E in a point P. You are going to see lots of drawings like this, of figures in space, and to learn to draw them yourself. You should examine them carefully to see how they work. We usually indicate a plane E by drawing a rectangle in E. Seen in perspective, the rectangle looks somewhat like a parallelogram. The line L punctures E at P. Part of L is dotted. This is the part that you "can't see", because the rectangular piece of E gets in the way. (For a discussion on drawing 3-dimensional figures see Appendix V.)

We have seen that two points determine a line. The next postulate specifies a similar determination of a plane.

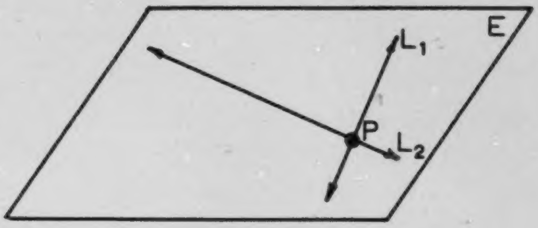
Postulate 7. Any three points lie in at least one plane, and any three non-collinear points lie in exactly one plane. More briefly, any three points are coplanar, and any three non-collinear points determine a plane.

Theorem 3-3. Given a line and a point not on the line, there is exactly one plane containing both of them.



The figure shows a plane E determined by the line L and the point P.

Theorem 3-4. Given two intersecting lines, there is exactly one plane containing them.



The figure shows two lines L_1 and L_2, intersecting in a point P. E is the plane that contains both lines.

Finally, we state one more postulate:

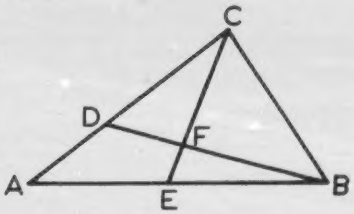
> Postulate 8. If two different planes intersect, then their intersection is a line.

Problem Set 3-1c

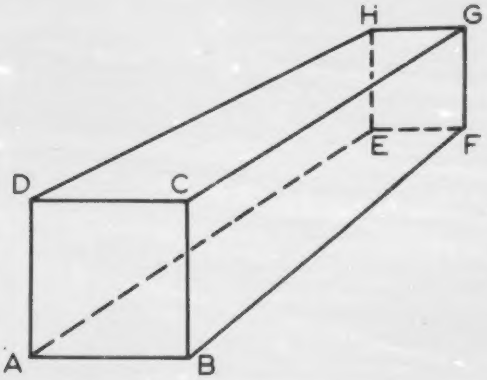
1. How many planes can contain one given point? two given points? three given points?

2. On a level floor, why will a four-legged table sometimes rock, while a three-legged table is always steady?
3. Complete: Two different lines may intersect in a __________, and two different planes may intersect in a __________.
4. Can two points be non-collinear? three points? four points? n points?
5. Write a careful definition of a set of non-collinear points.
6. Given: 1. Points A, B, C lie in plane E.
   2. Points A, B, C lie in plane F.

   Can you conclude that plane E is the same as plane F? Explain.
7. Complete the following statements using the accompanying diagram.
   a. Points D, C, and ___ are collinear.
   b. Points E, F, and ___ are collinear.
   c. Points B, ___, and A are collinear.
   d. Points A, B, C, D, E, F, are ____________________.



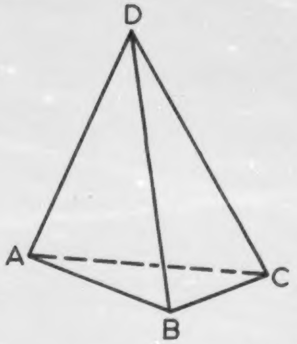
8. Examine the following figure of a rectangular solid until you see how it looks as a three-dimensional drawing. Then close the book and draw a figure like this for yourself. Practice until you are satisfied with the results.



9. After doing Problem 8, draw a figure that represents a cube.
10. Draw a plane E, using a parallelogram to indicate the plane.

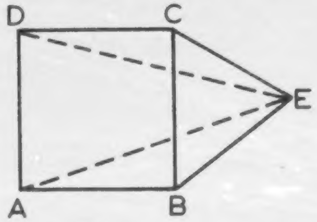
Draw a line segment which lies in the plane E. Draw a line that intersects the plane E but does not intersect the line segment. Use dashes to represent the part of the line hidden by the plane.

11. The accompanying figure is a triangular pyramid, or tetrahedron. It has four vertices: A, B, C, D, no three of which are collinear.



   a. Make a definition of an edge of this tetrahedron. Use the ideas of the text to help you form the definition.
   b. How many edges does the tetrahedron have? Name them.
   c. Are there any pairs of edges that do not intersect?
   d. A face is the triangular surface determined by any three vertices. There are four faces: ABC, ABD, ACD, BCD. Are there any pairs of faces that do not intersect? Explain.

12. How many different planes (determined by triplets of labeled points) are there in the pyramid shown? Make a complete list. (You should have seven planes.)



3-2. Theorems in the Form of Hypothesis and Conclusion.

Nearly every theorem is a statement that if a certain thing is true, then something else is also true. For example, Theorem 3-1 states that if L_1 and L_2 are two different lines, then L_1 intersects L_2 in at most one point. The if part of a theorem is called the hypothesis, or the given data, and the then part is called the conclusion, or the thing to be proved. Thus we can write Theorem 3-1 in this way:

Theorem 3-1. Hypothesis: L_1 and L_2 are two different lines.
Conclusion: L_1 intersects L_2 in at most one point.

Postulates, of course, are like theorems, except that they are not going to be proved. Most of them can be put in the same if ... then form as theorems. Postulate 1 can be stated this way:

Hypothesis: P and Q are two different points.

Conclusion: There is exactly one line containing P and Q.

There are cases in which the hypothesis-conclusion form does not seem natural or useful. For example, the second part of Postulate 5, expressed in this form, looks awkward:

Hypothesis: S is space.

Conclusion: Not all points of S are coplanar.

Such cases, however, are very rare.

It is not necessary, of course, that all theorems be stated in the hypothesis-conclusion form. It ought to be clear, regardless of the form in which the theorem is stated, what part of it is the hypothesis and what part is the conclusion. It is very important, however, that we be able to state a theorem in this form if we want to, because if we cannot, the chances are that we do not understand exactly what the theorem says.

Problem Set 3-2

1. Indicate which part of each of the following statements is the hypothesis and which part is the conclusion. If necessary, rewrite in if-then form first.
   a. If John is ill, he should see a doctor.
   b. A person with red hair is nice to know.
   c. Four points are collinear if they lie on one line.
   d. If I do my homework well, I will get a good grade.
   e. If a set of points lies in one plane, the points are coplanar.
   f. Two intersecting lines determine a plane.
2. Write the following statements in conditional, or if-then, form:
   a. Two different lines have at most one point in common.
   b. Every geometry student knows how to add integers.
   c. When it rains, it pours.
   d. A line and a point not on the line are contained in exactly one plane.

e. A dishonest practice is unethical.

f. Two parallel lines determine a plane.

3. Using the words "if" and "then", write Postulate 1 and Theorem 3-1 in conditional form. Indicate the hypothesis and the conclusion for each case.

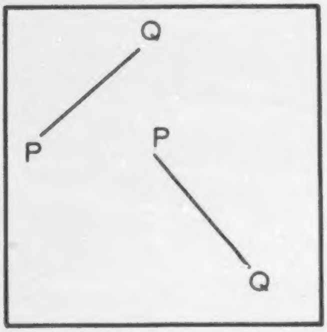
4. a. Does the following statement mean the same thing as Theorem 3-4? "Two lines always intersect in a point, and there is exactly one plane containing them." Why or why not?

   b. Write Theorem 3-4 in the "hypothesis and conclusion" form.

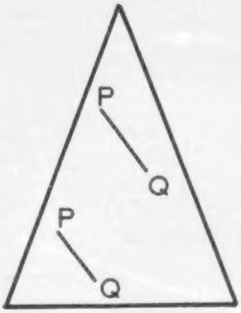
3-3. Convex Sets.

Definition. A set A is called convex if for every two points P and Q of A, the entire segment $\overline{PQ}$ lies in A.

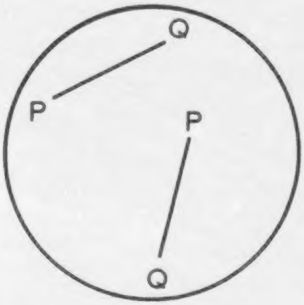
For example, the three sets pictured below are convex.



A

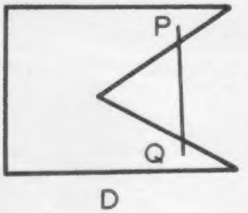


B

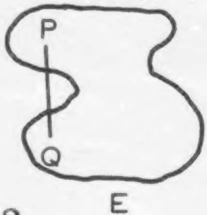


C

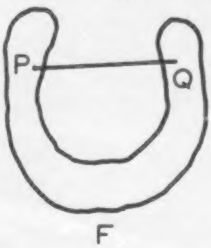
Here each of the sets A, B and C consists of a region of the plane. We have illustrated their convexity by showing a few segments $\overline{PQ}$. None of the sets D, E and F below is convex:



D



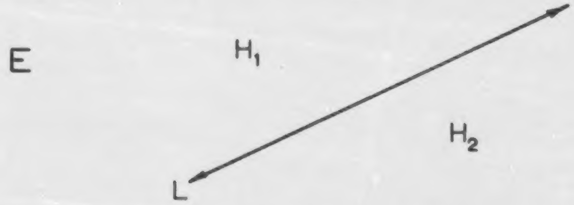
E



F

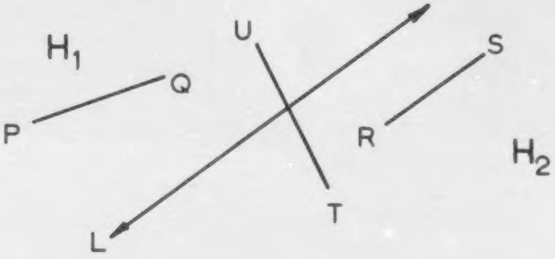
We have shown why not, by showing pairs of points P, Q for which the segment $\overline{PQ}$ does not lie entirely in the given set.

A convex set may be very large. For example, take a line L in a plane E and let H_1 and H_2 be the sets lying on the two sides of L, like this:



The two sets H_1 and H_2 are called half-planes or sides of L, and the line L is called an edge of each of them. (Notice that L does not lie in either of the two half-planes; L is not on either side of itself.)

If two points P and Q are in the same half-plane, say H_1, then the segment $\overline{PQ}$ also lies in H_1, and so does not intersect L.



Thus H_1 is convex. And in the same way, H_2 is convex; this is illustrated by the points R and S in the figure.

We notice, however, that if T and U are points belonging to different half-planes, then the segment $\overline{TU}$ intersects L, because you cannot get from one side of L to the other side without crossing the edge. We express this fact by saying that L separates H_1 from H_2 in the plane, or that L separates the plane into two half-planes H_1 and H_2.

This discussion is a fair account of the facts, but it is not very good mathematical form, because it is based on a postulate that we haven't even stated so far. We shall therefore state the postulate that is needed, and then state the definitions that are based on it.

Postulate 9. (The Plane Separation Postulate.) Given a line and a plane containing it. The points of the plane that do not lie on the line form two sets such that (1) each of the sets is convex and (2) if P is in one set and Q is in the other then the segment $\overline{PQ}$ intersects the line.

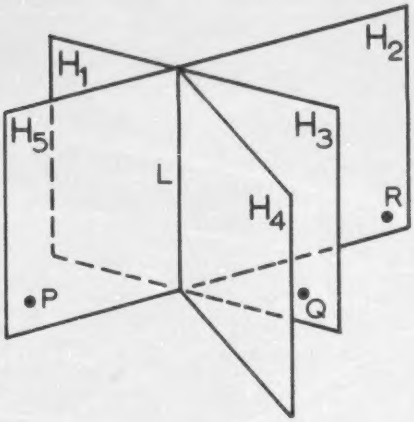
Definitions. Given a line L and a plane E containing it, the two sets determined by Postulate 9 are called half-planes, and L is called an edge of each of them. We say that L separates E into the two half-planes. If two points P and Q of E lie in the same half-plane, we say that they lie on the same side of L; if P lies in one of the half-planes and Q in the other they lie on opposite sides of L.

We see that the Plane Separation Postulate says two things about the two half-planes into which a line separates a plane:

(1) If two points lie in the same half-plane, then the segment between them lies in the same half-plane, and so never intersects the line.

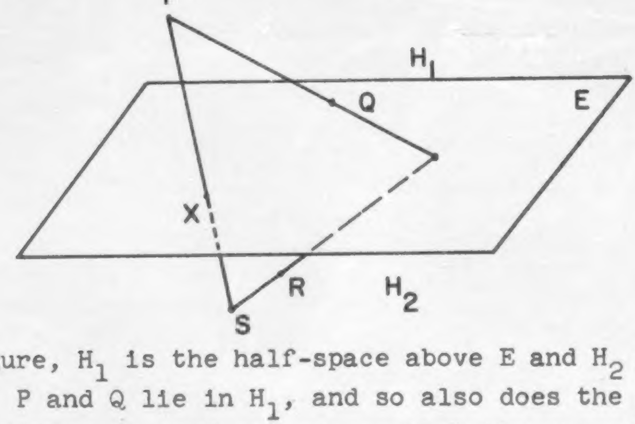
(2) If two points lie in different half-planes, then the segment between them always intersects the line.

If we do not restrict our attention to a single plane we can have many half-planes with the same edge. The picture



illustrates five of the infinitely many possible half-planes having line L for edge. Note that points P and Q, although they lie in different half-planes, cannot be said to be on opposite sides of L. This can only be said of points like P and R which are coplanar with L.

A plane separates space, in exactly the same way, into two convex sets called half-spaces.



In the figure, H_1 is the half-space above E and H_2 is the half-space below E. P and Q lie in H_1, and so also does the segment $\overline{PQ}$. P and S are in different half-spaces, so that the segment $\overline{PS}$ intersects E in a point X. R and S are in the same half-space H_2, and so also is the segment $\overline{RS}$.

This situation is described in the following postulate.

Postulate 10. (The Space Separation Postulate.) The points of space that do not lie in a given plane form two sets such that (1) each of the sets is convex and (2) if P is in one set and Q is in the other, then the segment $\overline{PQ}$ intersects the plane.

Definitions. The two sets determined by Postulate 10 are called half-spaces, and the given plane is called the face of each of them.

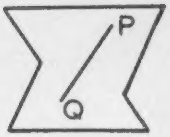
Note that while a line is an edge of infinitely many half-planes, a plane is a face of only two half-spaces.

Problem Set 3-3

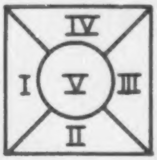
In answering the following questions use your intuitional understanding of planes and space in situations not covered by our postulational structure.

1. Be prepared to discuss the following questions orally.
   a. Is a line a convex set? Explain.
   b. Is a set consisting of only two points convex? Why?
   c. Is a ray a convex set?
   d. If one point is removed from a line, do the remaining points form a convex set? Why?
   e. Is the set of points on the surface of a sphere convex? Why?
   f. Is the space enclosed by a sphere a convex set?
   g. Does a point separate a plane? space? a line?
   h. Does a ray separate a plane? Does a line? Does a line segment?
   i. Can two lines in a plane separate the plane into two regions? Three regions? Four regions? Five regions?
   j. Into how many parts does a plane separate space? What are these parts called?

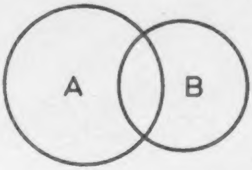
2. Every point on $\overline{PQ}$ is contained in the set shown. Does this mean that the set is convex? Explain.



3. Which of the regions indicated by Roman numerals are convex sets? Give reasons for your choice.



4. Is every plane a convex set? Explain. Which postulate is essential in your explanation?
5. The interiors of circles A and B are each convex sets.
   a. Is their intersection a convex set? Illustrate.
   b. Is their union a convex set? Illustrate.



6. If one point is removed from a plane, is the set formed convex? Why?
7. If L is a line in a plane E, is the set of all points of E on one side of L a convex set?
8. Draw a plane quadrilateral (a figure with four sides) whose interior is convex. Draw one whose interior is not convex.
9. Is the set of points containing all points on the surface and all points in the interior of a sphere convex?
10. Is the set of points in a torus (a doughnut shaped figure) convex?
11. Is the union of two half-planes which are contained in a plane the whole plane if
    a. the half-planes have the same edge? Explain.
    b. the edge of one half-plane intersects the edge of the other half-plane in exactly one point? Explain, using a diagram if necessary.
12. a. Into how many parts does a point on a line separate the line? What name would you suggest giving to each of these parts?
    b. Using the terminology you developed in part (a), write out a Line Separation Statement similar to Postulates 9 and 10.

13. How does a ray differ from a half-line?
14. Can three lines in a plane ever separate the plane into three regions? four regions? five regions? six regions? seven regions?
15. Into how many parts do two intersecting planes separate space? Two parallel planes?
16. What is the greatest number of parts into which space can be separated by three distinct planes? What is the least number?

*17. Write a careful explanation of why the following statement is true. The intersection of any two convex sets which have at least two points in common is convex. (Hint: Let P and Q be any two points belonging to the intersection.)

*18. Sketch any geometrical solid bounded by plane surfaces such that the set of points in the interior of the figure is not convex.

Review Problems

1. Each of 3 planes intersects each of the others. May they intersect in one line? Must all three intersect in one line? Explain.
2. How many planes will contain the three given points A, B, and C if no line contains them?
3. Write each of the following statements in the "if-then" form.
    a. Zebras with polka dots are dangerous.
    b. Rectangles whose sides have equal lengths are squares.
    c. There will be a celebration if Oklahoma wins.
    d. A plane is determined by any two intersecting lines.
    e. Cocker spaniel dogs are sweet tempered.
4. Supply the following information about the postulates in the chapter.

    What property of each of the half-planes is mentioned in the Plane Separation Postulate?

    Do the half-spaces of the Space Separation Postulate have the same property?

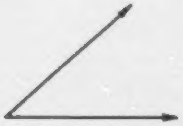
5. Criticize the following statement:
   "The top of the table is a plane."
6. List all the situations we have studied which determine a single plane.
7. A set is convex if for every pair of points in it, all points of the segment joining the two points lie ____________________
   ____.
8. Given that plane E separates space into half-spaces R and S, and that point A is in R and point B is in S, does $\overline{AB}$ have to intersect E?
9. L_1 intersects plane E in P but does not lie in E. L_2 lies in plane E but does not contain point P. Is it possible for L_1 and L_2 to intersect? Explain.
10. a. A set of points is collinear if ____________________
    ____.
    b. A set of points is coplanar if ____________________
    ____.
    c. May 5 points be collinear?
    d. Must 2 points be collinear?
    e. May n points be collinear?
    f. Must 5 points be coplanar?
    g. May n points be coplanar?
11. Points P and Q lie in both planes E and F which intersect in line $\overleftrightarrow{AB}$. Would it be correct to say that P and Q lie on $\overleftrightarrow{AB}$? Explain.
12. Is the union of a half-plane and a ray on its edge convex?

Chapter 4

ANGLES AND TRIANGLES

4-1. *The Basic Definitions*.

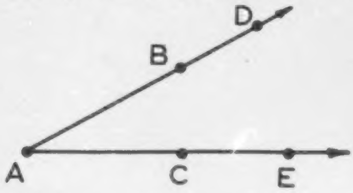
An angle is a figure that looks like one of these:



To be more exact:

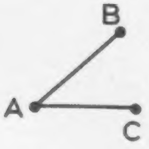
Definitions. An *angle* is the union of two rays which have the same end-point but do not lie in the same line. The two rays are called the *sides* of the angle, and their common end-point is called the *vertex*.

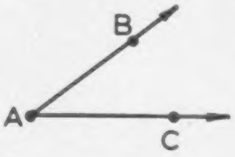
The angle which is the union of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is denoted by $\angle BAC$, or by $\angle CAB$, or simply by $\angle A$ if it is clear which rays are meant. Notice that $\angle BAC$ can be equally well described by means of A and any two points on different sides of the angle.



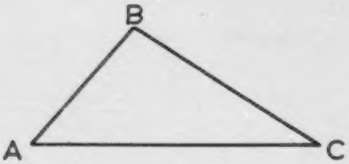
In the above figure $\angle DAE$ is the same as $\angle BAC$, because $\overrightarrow{AD}$ is the same as $\overrightarrow{AB}$ and $\overrightarrow{AE}$ is the same as $\overrightarrow{AC}$.

Notice that an angle goes out infinitely far in two directions, because its sides are rays, rather than segments. The figure on the left, below, determines an angle uniquely, but is not all of the angle; to get all of the angle, we have to extend the segments $\overline{AB}$ and $\overline{AC}$ getting rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$, as on the right.



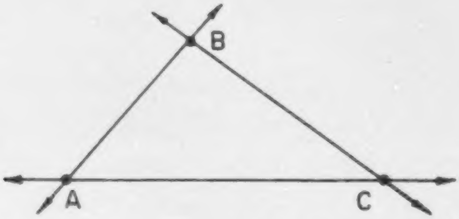


Definitions. If A, B, and C are any three non-collinear points, then the union of the segments $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ is called a triangle,



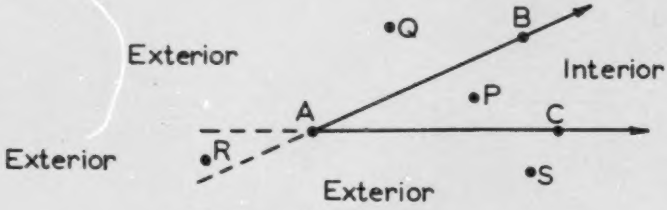
and is denoted by $\triangle ABC$; the points A, B and C are called its vertices, and the segments $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ are called its sides. Every triangle determines three angles; $\triangle ABC$ determines the angles $\angle BAC$, $\angle ABC$ and $\angle ACB$, which are called the angles of $\triangle ABC$. For short, we will often write them simply as $\angle A$, $\angle B$, and $\angle C$.

Note that while $\triangle ABC$ determines these three angles, it does not actually contain them. Just as a school does not contain its own graduates, so a triangle does not contain its own angles, because the sides of a triangle are segments, and the sides of an angle are rays. To draw the angles of a triangle, we would have to extend the sides of the triangle to get rays, like this:



There usually is not much point in doing this, however, because it is plain what the angles of a triangle are supposed to be.

The interior of an angle consists of all points that lie inside the angle; and the exterior of an angle consists of all the points that lie outside, like this:

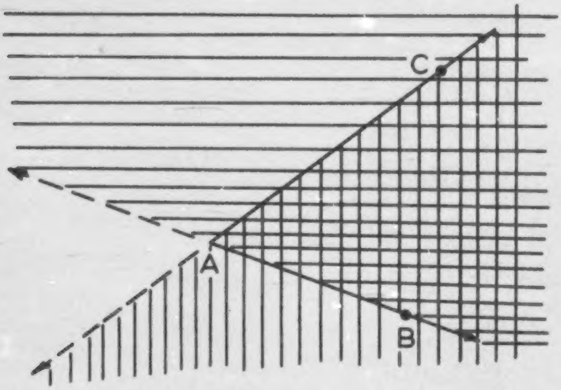


We can state this more exactly as follows:

Definitions. Let $\angle$BAC be an angle lying in plane E. A point P of E lies in the interior of $\angle$BAC if (1) P and B are on the same side of the line $\overleftrightarrow{AC}$ and also (2) P and C are on the same side of the line $\overleftrightarrow{AB}$. The exterior of $\angle$BAC is the set of all points of E that do not lie in the interior and do not lie on the angle itself.

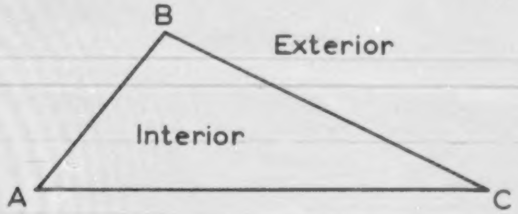
You should check carefully to make sure that this really says what we want it to say. In the figure, P is in the interior, because P and B are on the same side of $\overleftrightarrow{AC}$ and also P and C are on the same side of $\overleftrightarrow{AB}$. Q is in the exterior, because Q and C are not on the same side of $\overleftrightarrow{AB}$. R is in the exterior, because R is on the "wrong side" of both of the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$. S is in the exterior because it is on the "wrong side" of $\overleftrightarrow{AC}$.

Notice that we have defined the interior of an angle as the intersection of two half-planes. The half-planes look like this:



Here one of the half-planes is cross-hatched horizontally, the other is cross-hatched vertically, and the interior of $\angle$BAC is cross-hatched both ways.

The interior of a triangle consists of the points that lie inside it, like this:



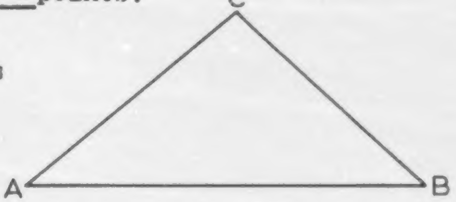
More precisely:

Definitions. A point lies in the interior of a triangle if it lies in the interior of each of the angles of the triangle. A point lies in the exterior of a triangle if it lies in the plane of the triangle but is not a point of the triangle or of its interior.

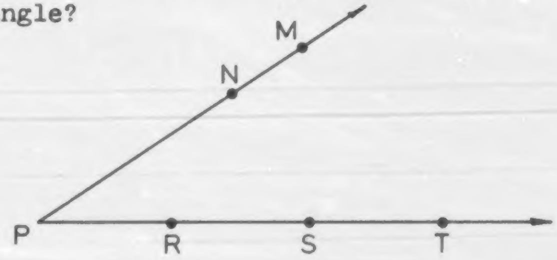
You should check carefully to make sure that this really says what we want it to say.

Problem Set 4-1

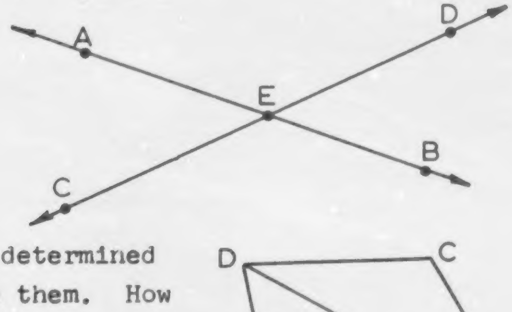
1. Complete this definition of angle: An angle is the ______________ of two ___________ which have the same endpoint, but do not lie in the same __________________.
2. Complete this definition of triangle: A triangle is the _______________ of the three _____________ joining each pair of three _______________ points.
3. Are the sides $\overline{AC}$ and $\overline{AB}$ of △ABC the same as the sides of ∠A? Explain.



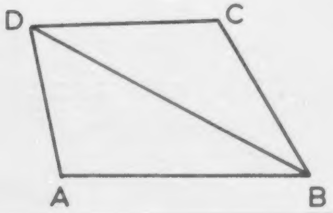
4. Is the union of two of the angles of a triangle the same as the triangle itself? Why?
5. Into how many regions do the angles of a triangle separate the plane of the triangle?
6. Complete:
   ∠P = ∠NPS = ∠MPR
   = _______ = _______
   = _______ = _______.



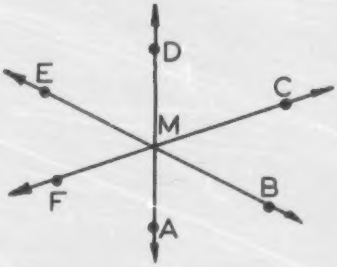
7. Name the angles in the figure.



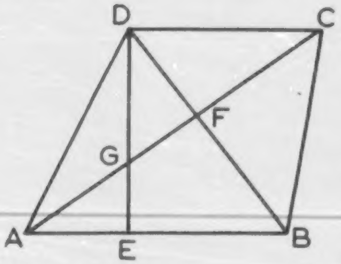
8. How many angles are determined by the figure? Name them. How many may be named using the vertex letter only?



9. Name the angles in the figure. (There are more than six.)

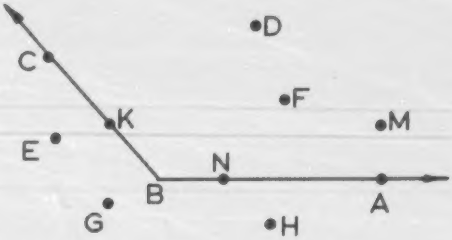


10. Name all the triangles in the figure.(There are more than eight).



11. a. Name the points of the figure which are in the interior of ∠CBA.

    b. Name the points of the figure in the exterior of ∠B.



12. Is the vertex of an angle in the interior of the angle? in the exterior? Explain.

13. Is the interior of an angle a convex set? is the exterior?

14. Is a triangle a convex set?

15. Is the interior of a triangle a convex set? is the exterior?

16. a. Can a point be in the exterior of a triangle and in the interior of an angle of the triangle? Illustrate.

    b. Can a point be in the exterior of a triangle and not in the interior of any angle of the triangle? Illustrate.

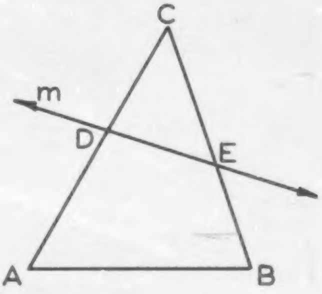
17. Given △ABC, and a point P. P is in the interior of ∠BAC and also in the interior of ∠ACB. What can you conclude about point P?

18. Given △ABC and a point P. P and A are on the same side of $\overleftrightarrow{BC}$. P and B are on the same side of $\overleftrightarrow{AC}$.

a. Is P in the interior of $\angle ACB$?

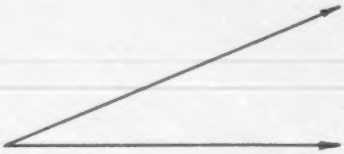
b. Is P in the interior of $\triangle ABC$?

19. Carefully explain why the following statement is true: If a line m intersects two sides of a triangle ABC in points D and E, not the vertices of the triangle, then line m does not intersect the third side. (Hint: Show that A and B are in the same half-plane.)



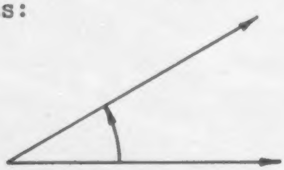
4-2. Remarks On Angles.

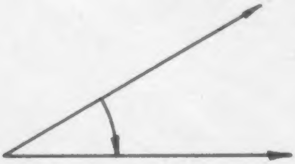
What we have presented in this chapter is the simplest form of the idea of an angle. According to our definition, an angle is simply a set which is the union of two non-collinear rays, like this:



Angles, in this sense, will be quite good enough for the purposes of this course. Later, you will see the idea of an angle in various other forms. Here we explain these other forms briefly, merely in order to avoid confusion in case you may have heard of them already.

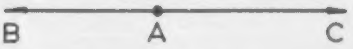
(1) In the first place, we sometimes think of an angle as being obtained by rotating a ray from one position to another. In this case, one ray is the initial side, and the other is the terminal side. Thus we would consider the two angles below as being different, because the rotations are in two different directions:





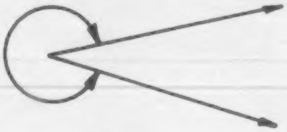
The first is called a positive angle; the rotation is counterclockwise. The second is a negative angle; the rotation is clockwise.

(2) People sometimes speak of straight angles, which look like this:



Here the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are considered to form an angle, even though A, B, and C are collinear.

(3) Finally, we sometimes distinguish between an ordinary angle and a reflex angle having the same rays as its sides. The double-headed arrow below is supposed to indicate a reflex angle:



These complications, and various others of the same sort, will not be used in this book, because they will not be needed. For example, the angles of a triangle are never reflex angles, and there is no reasonable way to decide in which direction they should be considered to go. Not until we get to trigonometry do these fancy angles become necessary and important.

4-3. Measurement Of Angles.

Angles are usually measured in degrees, with a protractor. With the protractor placed as in the figure below, with its edge on the edge of the half-plane H, we can read off the measures of a large number of angles.

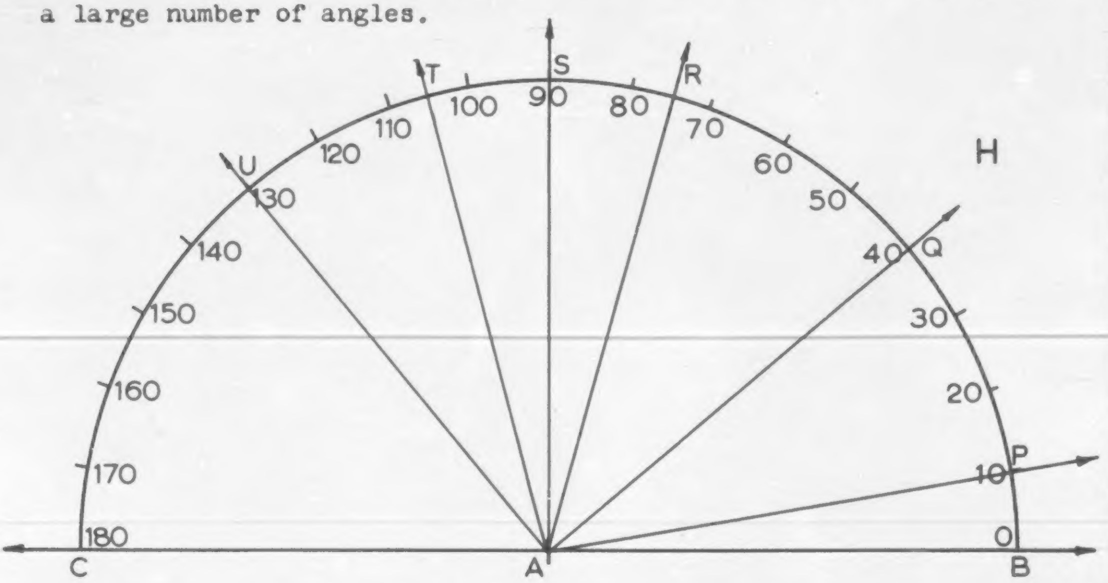


Figure A.

The number of degrees in an angle is called its measure. If there are r degrees in the angle $\angle XAY$, then we write

$$m \angle XAY = r.$$

For example, in the figure we read off that

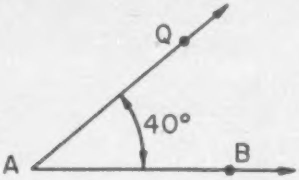
$$m \angle PAB = 10,$$
$$m \angle QAB = 40,$$
$$m \angle RAB = 75,$$
$$m \angle SAB = 90,$$
$$m \angle TAB = 105,$$

and so on. Of course, the rays that are drawn form more angles than this. By subtraction, we can see that

$$m \angle QAP = 40 - 10 = 30,$$
$$m \angle SAR = 90 - 75 = 15,$$

and so on.

Since $m\angle QAB = 40$, we speak of $\angle QAB$ as a 40° angle, and we indicate its measure in a figure like this:



But we don't need to use the degree sign when we write $m\angle QAB = 40$, because as we explained at the outset, $m\angle QAB$ means the number of degrees in the angle.

Notice that in Figure A there is no such thing as the angle $\angle CAB$, because the rays $\overrightarrow{AC}$ and $\overrightarrow{AB}$ are collinear. But we notice that the ray $\overrightarrow{AC}$ corresponds to the number 180 on the number-scale of the protractor, and the ray $\overrightarrow{AB}$ corresponds to the number 0. Therefore we can find $m\angle CAU$ by writing

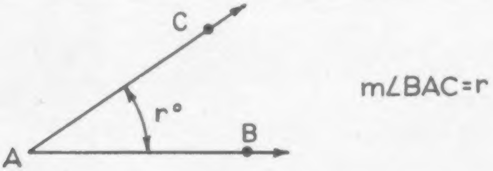
$$\begin{aligned} m\angle CAU &= 180 - 130, \\ &= 50. \end{aligned}$$

Similarly,

$$\begin{aligned} m\angle CAQ &= 180 - 40, \\ &= 140. \end{aligned}$$

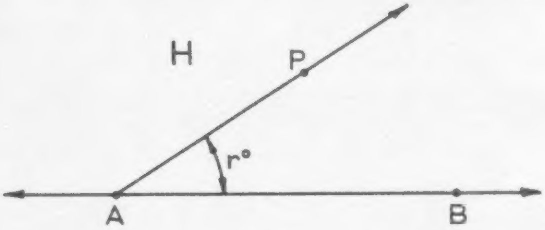
The following postulates merely summarize the facts about protractors that we have just been discussing. Each of them is illustrated by a figure.

Postulate 11. (The Angle Measurement Postulate.) To every angle $\angle BAC$ there corresponds a real number between 0 and 180.

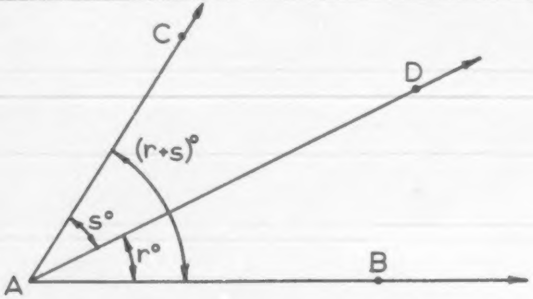


Definition. The number specified by Postulate 11 is called the measure of the angle, and written as $m\angle BAC$.

Postulate 12. (The Angle Construction Postulate.) Let $\overrightarrow{AB}$ be a ray on the edge of the half-plane H. For every number r between 0 and 180 there is exactly one ray $\overrightarrow{AP}$, with P in H, such that m $\angle$ PAB = r.

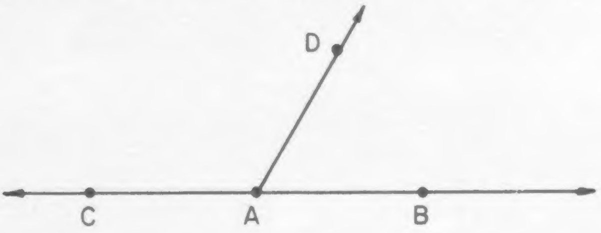


Postulate 13. (The Angle Addition Postulate.) If D is a point in the interior of $\angle$ BAC, then m $\angle$ BAC = m $\angle$ BAD + m $\angle$ DAC.



In was on this basis that we computed the measures of angles by subtraction, with a protractor placed with its edge on the ray $\overrightarrow{AB}$. (m$\angle$ DAC = m$\angle$ BAC - m$\angle$ BAD.)

Two angles form a linear pair if they look like this:



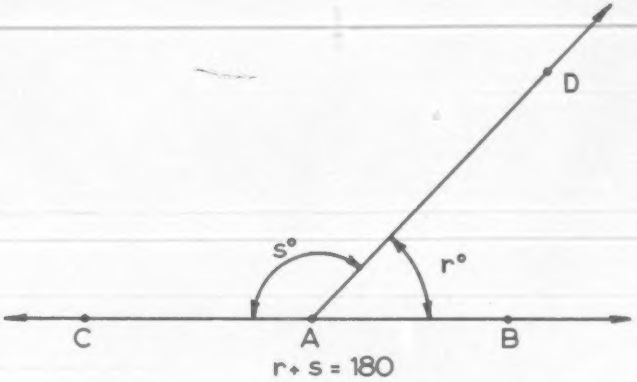
That is:

Definition. If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays, and $\overrightarrow{AD}$ is another ray, then $\angle BAD$ and $\angle DAC$ form a linear pair.

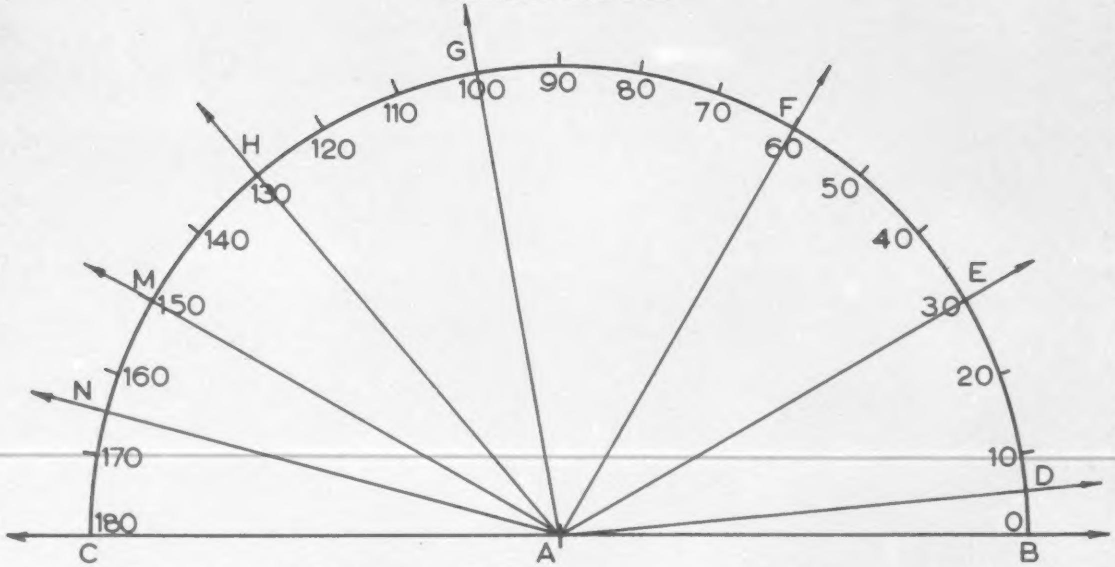
Definition. If the sum of the measures of two angles is 180, then the angles are called supplementary, and each is called a supplement of the other.

Hence the name of the following postulate.

Postulate 14. (The Supplement Postulate.) If two angles form a linear pair, then they are supplementary.



Problem Set 4-3

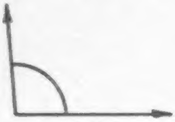

G
H
F
M
E
N
D
C
A
B
0
10
20
30
40
50
60
70
80
90
100
110
120
130
140
150
160
170
180

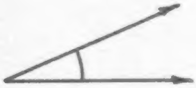
1. Using the figure, find the value of each of the following:

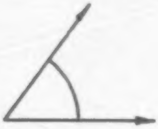
   a. $m\angle FAB$.
   b. $m\angle EAB$.
   c. $m\angle MAC$.
   d. $m\angle FAE$.
   e. $m\angle GAE$.
   f. $m\angle MAN$.
   g. $m\angle EAD$.
   h. $m\angle FAG + m\angle GAH$.
   i. $m\angle GAF + m\angle FAE$.
   j. $m\angle MAB - m\angle FAB$
   k. $m\angle HAB - m\angle DAB$.
   l. $m\angle NAE - m\angle NAH$.

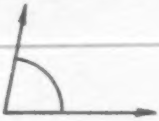
2. With continued practice you should be able to estimate the size of angles fairly accurately without using a protractor. Do not use a protractor to decide which of the angles shown have measures within the indicated ranges.

Match the corresponding pairs:

a.

m. $15 < x < 35$.

b.

n. $70 < x < 90$.

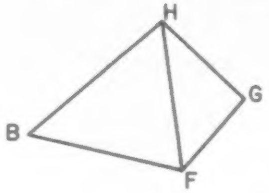
c.

p. $80 < x < 100$.

d.

q. $45 < x < 60$.

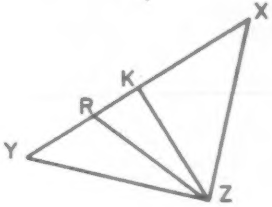
3. Using only a straightedge and not a protractor, sketch angles whose measures are approximately 30, 150, 45, 60, 135, 90. Then use your protractor to check your sketches.

4. On the edge of a half-plane, take a segment $\overline{AB}$ about 3 inches long. At A draw ray $\overrightarrow{AC}$ in the half-plane forming $\angle BAC$ of 58°. At B draw ray $\overrightarrow{BD}$ in the same half-plane forming $\angle ABD$ of 72°. Measure the remaining angle of the triangle formed.

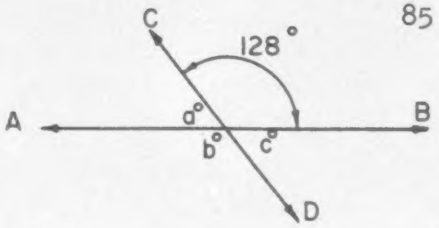
5. In the figure,
   a. $m\angle BHF + m\angle GHF = m\angle$ ____?____
   b. $m\angle GFH + m\angle BFH = m\angle$ ____?____



6. In the figure,
   a. $m\angle XZK + m\angle KZR + m\angle YZR = m\angle$ ____?____
   b. $m\angle XZR - m\angle RZK = m\angle$ ___?___
   c. $m\angle XZY - m\angle XZK = m\angle$ ___?___
   d. If Y, R, K and X are collinear, then $m\angle YRZ + m\angle ZRX =$ ___?___



7. In the figure, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect forming four angles. Using the indicated measure, find a, b and c.

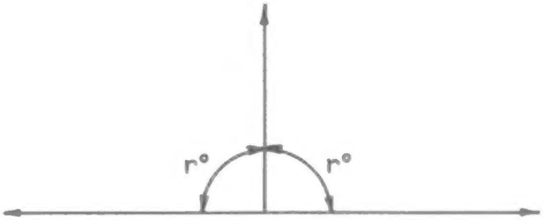


8. Determine the supplement of each of the following: 110°, 90°, 36°, 15.5°, n°, $(180 - n)^\circ$, $(90 - n)^\circ$.
9. If one of two supplementary angles has a measure 30 more than the measure of the other, what is the measure of each angle?
10. If the measure of an angle is twice the measure of its supplement, find the measure of the angle.
11. The measure of an angle is four times the measure of its supplement. Find the measure of each angle.
12. a. Given a ray $\overrightarrow{AC}$ lying on the edge of a half-plane H, and a number r between 0 and 180. In how many ways can you construct a ray $\overrightarrow{AB}$ in H such that $m\angle BAC = r$? Why?

    b. Given a ray $\overrightarrow{AC}$ lying in a plane E, and a number r between 0 and 180. In how many ways can you construct a ray $\overrightarrow{AB}$ in E such that $m\angle BAC = r$? Why?

4-4. Perpendicularity, Right Angles, and Congruence of Angles.

Definitions. If the two angles of a linear pair have the same measure, then each of the angles is a right angle.



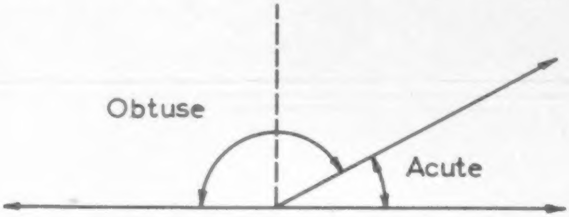
Since $r + r = 180$, by the Supplement Postulate, we see that *a right angle is an angle of 90°*. This can be regarded as an alternative definition of a right angle; it is equivalent to our first definition.

In terms of right angles it is easy to define perpendicularity of any combination of line, ray or segment. In applying the following definition remember that a ray or a segment determines a unique line which contains it.

Definition. Two intersecting sets, each of which is either a line, a ray or a segment, are *perpendicular* if the two lines which contain them determine a right angle.

Definition. If the sum of the measures of two angles is 90, then the angles are called *complementary*, and each of them is called a *complement* of the other. (Compare this with the definition of *supplementary angles*, just before the statement of the Supplement Postulate.)

An angle with measure less than 90 is called *acute*, and an angle with measure greater than 90 is called *obtuse*.

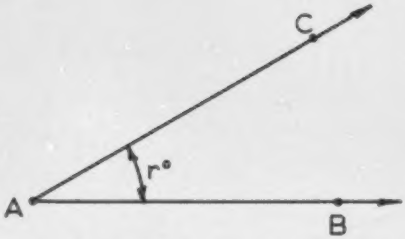


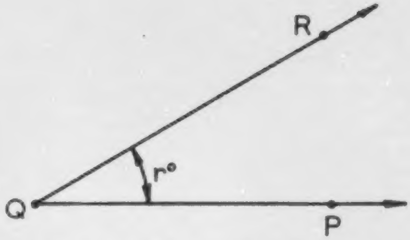
Definition. Angles with the same measure are called *congruent angles*.

That is, $\angle BAC$ and $\angle PQR$ are congruent if $m\angle BAC = m\angle PQR$. In this case we write

$$\angle BAC \cong \angle PQR.$$

Notice that the equation $m\angle BAC = m\angle PQR$ and the congruence $\angle BAC \cong \angle PQR$ are completely equivalent; we can replace one by the other any time we want to.





The following theorems are easy to prove, if we remember clearly what the words mean:

Theorem 4-1. If two angles are complementary, then both of them are acute.

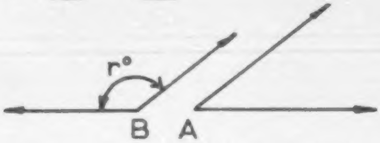
Theorem 4-2. Every angle is congruent to itself.

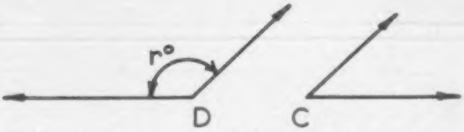
Theorem 4-3. Any two right angles are congruent.

Theorem 4-4. If two angles are both congruent and supplementary, then each of them is a right angle.
(Hint: Let r be the number which is the measure of each of the two angles, and then find out what r must be.)

Theorem 4-5. Supplements of congruent angles are congruent.

Restatement: If (1) $\angle B \cong \angle D$, (2) $\angle A$ and $\angle B$ are supplementary and (3) $\angle C$ and $\angle D$ are supplementary, then (4) $\angle A \cong \angle C$.





Proof: The statement that $\angle B \cong \angle D$ means that $m\angle B$ and $m\angle D$ are the same number r, as in the figure. Since $\angle A$ and $\angle B$ are supplementary, it follows that

$$m\angle A = 180 - m\angle B = 180 - r.$$

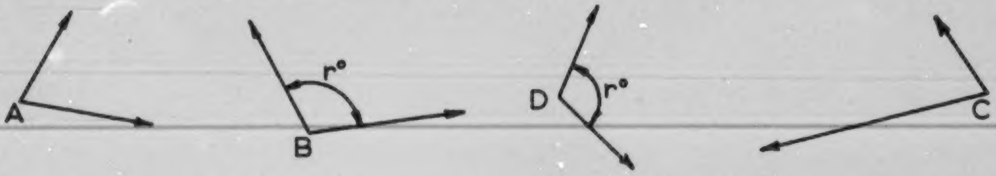
For the same reason,

$$m\angle C = 180 - m\angle D = 180 - r.$$

Therefore $m\angle A = m\angle C$, which means that $\angle A \cong \angle C$.

You must not conclude from the above picture that supplementary angles must necessarily be placed beside one another in a way that makes it evident that their measures add up to 180.

The following picture also serves to illustrate Theorem 4-5.

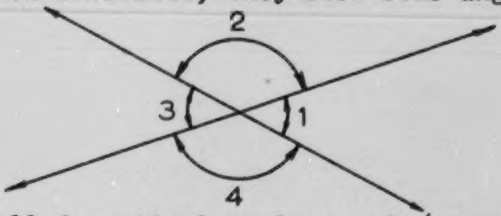


In drawing pictures to illustrate theorems or problems you should realize that the figure in the book is not the only correct one, and you should try to make your picture different from the one given in the book.

Theorem 4-6. Complements of congruent angles are congruent.

The proof of the theorem is exactly analogous to the preceding proof, and you should write it out for yourself.

When two lines intersect, they form four angles, like this:

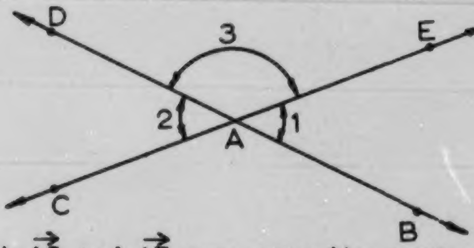


$\angle 1$ and $\angle 3$ are called vertical angles, and $\angle 2$ and $\angle 4$ are also called vertical angles. More precisely:

Definition. Two angles are vertical angles if their sides form two pairs of opposite rays.

It looks as if these pairs of vertical angles ought to be congruent, and in fact this is what always happens:

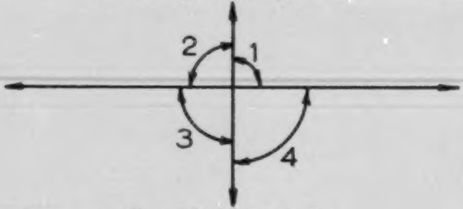
Theorem 4-7. Vertical angles are congruent.



Proof: Given that $\overrightarrow{AC}$ and $\overrightarrow{AE}$ are opposite rays, and $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are opposite rays, so that $\angle 1$ and $\angle 2$ are vertical angles. Then $\angle 1$ and $\angle 3$ are supplementary, and $\angle 2$ and $\angle 3$ are supplementary.

Since $\angle 3$ is congruent to itself, this means that $\angle 1$ and $\angle 2$ have congruent supplements. By Theorem 4-5, $\angle 1 \cong \angle 2$, which was to be proved.

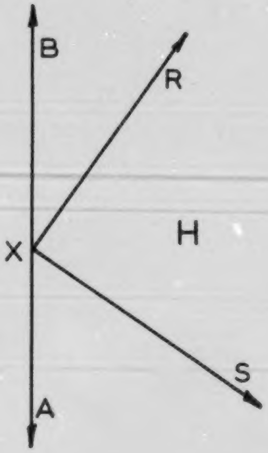
Theorem 4-8. If two intersecting lines form one right angle, then they form four right angles.



You should be able to supply the proof.

Problem Set 4-4

1. a. In a plane, how many perpendiculars can be drawn to a line at a given point on the line?
   b. In space, how many perpendiculars can be drawn to a line at a given point on the line?
2. If $\overrightarrow{OR}$ and $\overrightarrow{OS}$ are opposite rays and $\overrightarrow{ON}$ is a ray such that $m\angle RON = m\angle SON$, what can you conclude about $\overleftrightarrow{ON}$ and $\overleftrightarrow{RS}$? Explain.
3. In half-plane H, $\overrightarrow{XB}$ and $\overrightarrow{XA}$ are opposite rays, $m\angle RXB = 35$ and $m\angle RXS = 90$.
   a. Name a pair of perpendicular rays, if any occur in the figure.
   b. Name a pair of complementary angles, if any occur in the figure.
   c. Name a pair of vertical angles, if any occur in the figure.
   d. Name two pairs of supplementary angles in the figure.



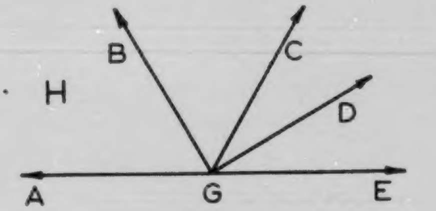
4. Determine the measure of a complementary angle for each of the following:
   a. 10°.
   b. 80°.
   c. 44.5°.
   d. x°.
   e. $(90 - x)^\circ$.
   f. $(180 - x)^\circ$.

5. a. If two angles with the same measure are supplementary, what is the measure of each?
   b. If two angles with the same measure are complementary, what is the measure of each?

6. a. If two lines intersect, how many pairs of vertical angles are formed?
   b. If the measure of any one of the angles in (a) is 70, what is the measure of each of the others?
   c. If all of the angles in (a) are congruent, what is the measure of each?

7. If one of a pair of vertical angles has a measure of r, write the formulas for the measures of the other three angles formed.

8. In half-plane H, $\overrightarrow{GE}$ and $\overrightarrow{GA}$ are opposite rays,
   $m\angle AGB = m\angle BGC$, and
   $m\angle CGD = m\angle DGE$.
   Find $m\angle BGD$.



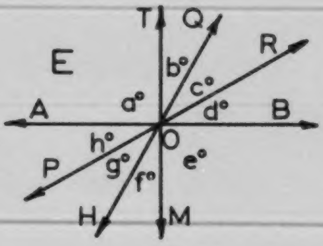
9. Prove Theorem 4-1.

10. Prove Theorem 4-4.

11. Given: In the figure for Problem 8, $\overrightarrow{GB} \perp \overrightarrow{GD}$ and $\overrightarrow{GA}$ and $\overrightarrow{GE}$ are opposite rays.
    Prove: $\angle AGB$ and $\angle DGE$ are complementary.

12. Given: In plane E, lines $\overleftrightarrow{AB}$, $\overleftrightarrow{PR}$, $\overleftrightarrow{HQ}$, $\overleftrightarrow{MT}$ intersect at O. $\overleftrightarrow{TM} \perp \overleftrightarrow{AB}$.

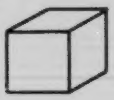
Prove: $b + g + d = a$.

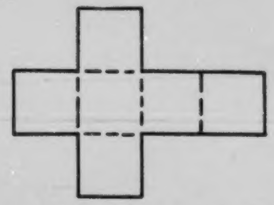


13. If $\overrightarrow{OA}$ and $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are three different rays in a plane, no two of them opposite, indicate true or false for each of the following statements and explain your answer.
    a. $m\angle AOB + m\angle BOC = m\angle AOC$.
    b. $m\angle AOB + m\angle BOC + m\angle AOC = 360$.

14. The measure of an angle is nine times that of its supplement. What is the measure of the angle?

15. A layout drawing is a plane drawing which can be folded to form the boundary of a given solid. Below is pictured a cube and a layout drawing for it.

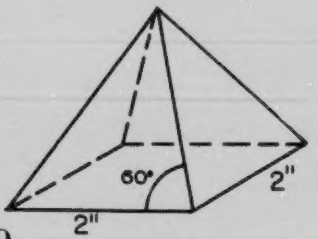




(Dotted lines indicate folds.)

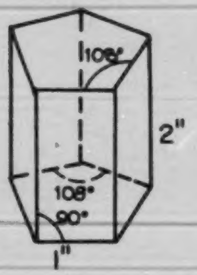
Use your imagination, your ruler and your protractor to make a layout drawing for each of the figures below. Then cut out your drawing, fold on dotted lines, and tape together. Use cardboard or heavy paper for a rigid figure.

a. A pyramid whose base is a square with 2" sides and whose other faces are isosceles triangles with 60° base angles.



(Problem 15 continued)

b. A prism whose bases are pentagons with 1 inch sides and 108° angles, and whose height is 2 inches.



Review Problems

1. What tool is used to measure angles?
2. To every angle there corresponds a real number between __________ and __________, called the measure of the angle.
3. An angle with a measure of less than 90 is __________.
4. Two angles formed by the union of two opposite rays and a third ray all with the same end point are a ________ ________ of angles.
5. If the sum of the measure of two angles is 90, then each is called a _______ of the other.
6. An angle with a measure greater than 90 is called _______.
7. Angles with the same measure are _______.
8. If two angles are both congruent and supplementary, then each of them is a ____________.
9. Supplements of congruent angles are ____________.
10. If two angles are complementary, then each of them is ____________.
11. An angle is the _______ of two _______ which have a common end point.
12. If X, Y, Z are three _______ points, the union of the three segments connecting them in pairs is a _______.
13. A point X is in the interior of $\angle RST$ if points R and _______ lie on the same side of $\overleftrightarrow{ST}$ and if points X and _______ lie on the same side of _______.
14. If the sum of the measures of two angles is _______ they are

called complementary and if the sum is ______ they are called ________.

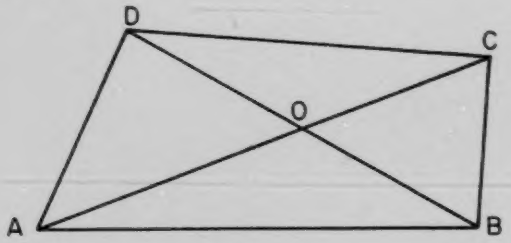
15. Two opposite angles formed by two intersecting lines are ________ angles. They are always congruent.

16. $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays. The points E, F, and H are on the same side of $\overleftrightarrow{AB}$. Points E and H are on opposite sides of $\overleftrightarrow{BF}$. Points A and H are on the same side of $\overleftrightarrow{BF}$. $\overrightarrow{BF}$ is $\perp$ $\overleftrightarrow{AC}$ and $\overrightarrow{BE} \perp \overrightarrow{BH}$. $m\angle FBE = 20$. Draw the figure and find:

a. $m\angle EBA$. b. $m\angle FBH$. c. $m\angle EBC$.

17.

Given:	Find:
$m\angle BCD = 90$,	a. $m\angle DOC$.
$m\angle BOC = 50$,	b. $m\angle BCO$.
$m\angle DCO = 25$,	c. $m\angle DOA$.
$m\angle DAO = 45$.	d. $m\angle AOB$.



18. If one of two supplementary angles has a measure of 50 more than the measure of the other, what is the measure of each angle?

19. The measure of an angle is five times that of its complement. Find the measure of each angle.

20. Under what conditions are the angles of a linear pair congruent?

21. Is there a point in the plane of a triangle such that the point is neither in the exterior nor the interior of a triangle and neither in the interior nor the exterior of any of its angles?

22. Is the measure of an angle added to the measure of an angle the measure of an angle? Explain.

23. Could the interior of a triangle be considered as the intersection of three half planes? Illustrate.

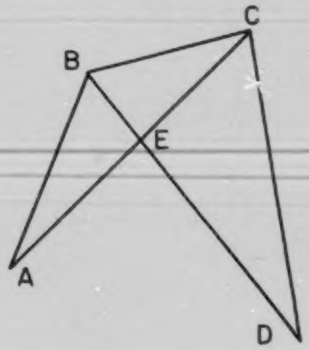
24. How many triangles are in this figure?

25. Does $m \angle BAC = m \angle BAE$?

26. Does $\angle BAC = \angle BAE$?

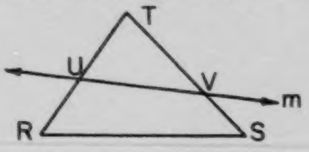
27. Is $\angle ABE$ supplementary to $\angle EBC$?

28. How many angles are indicated in the drawing?

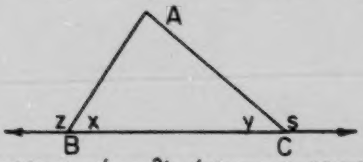


Problems 24 - 28.

29. Explain carefully why the following statement is true: If a line m intersects 2 sides of a triangle $\triangle RST$ in points U and V, not the vertices of the triangle, then line m does not intersect the third side.



30. Given in the figure $\angle x \cong \angle y$. Prove: $\angle z \cong \angle s$.



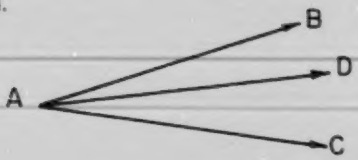
31. If you were given that $\angle a \cong \angle b$ and that $\angle x$ is supplementary to $\angle a$ and that $\angle y$ is supplementary to $\angle b$, what theorem or postulate would you use to prove that $\angle x \cong \angle y$?

32. The Angle Measurement Postulate places what limitation on angle measures?

33. Is the following a correct restatement of the Angle Construction Postulate: Given a ray $\overrightarrow{XY}$ and a number k between 0 and 180 there is exactly one ray $\overrightarrow{XP}$ such that $m \angle PXY = k$? Explain.

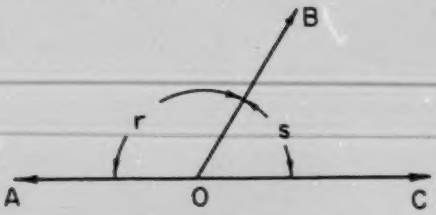
34. By giving its name, or by stating it in full, give the postulate which seems to you to be most appropriate in each of the following cases, as a reason for the statement.

a.



$m\angle DAC = m\angle BAC - m\angle BAD.$

b.



$r + s = 180.$

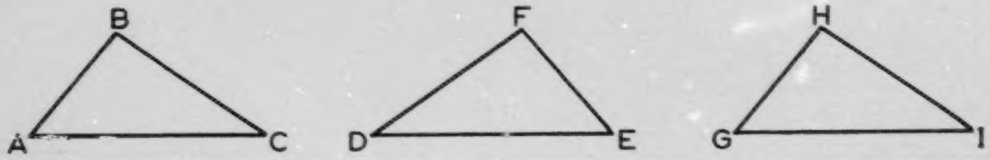
35. Is the following statement always true?
If $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at O, then $\angle AOC \cong \angle BOD$.

Chapter 5

CONGRUENCES

5-1. The Idea of a Congruence.

Roughly speaking, two geometric figures are congruent if they have exactly the same size and shape. For example, in the figure below, all three triangles are congruent.



One way of describing the situation is to say that any one of these triangles can be moved onto any other one, in such a way that it fits exactly. Thus, to show what we mean by saying that two triangles are congruent, we have to explain what points are supposed to go where. For example, to move $\triangle ABC$ onto $\triangle DFE$, we should put A on E, B on F, and C on D. We can write down the pairs of corresponding vertices like this:

$$A \longleftrightarrow E$$
$$B \longleftrightarrow F$$
$$C \longleftrightarrow D.$$

To describe the congruence of the first triangle and the third, we should match up the vertices like this:

$$A \longleftrightarrow G$$
$$B \longleftrightarrow H$$
$$C \longleftrightarrow I.$$

How would you match up the vertices to describe the congruence of the second triangle with the third?

A matching-up scheme of this kind is called a one-to-one correspondence between the vertices of the two triangles. If the matching-up scheme can be made to work -- that is, if the triangles can be made to fit when the vertices are matched up in the prescribed way -- then the one-to-one correspondence is called a *congruence* between the two triangles. For example, the correspondences that we have just given are congruences. On the other hand, if we write

$$\begin{aligned} A &\leftrightarrow F \\ B &\leftrightarrow D \\ C &\leftrightarrow E, \end{aligned}$$

this does give us a one-to-one correspondence, but does *not* give us a congruence, because the first and second triangles cannot be made to coincide by this particular matching-up scheme.

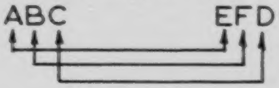
We can write down one-to-one correspondences more briefly, in one line. For example, the correspondence

$$\begin{aligned} A &\leftrightarrow E \\ B &\leftrightarrow F \\ C &\leftrightarrow D, \end{aligned}$$

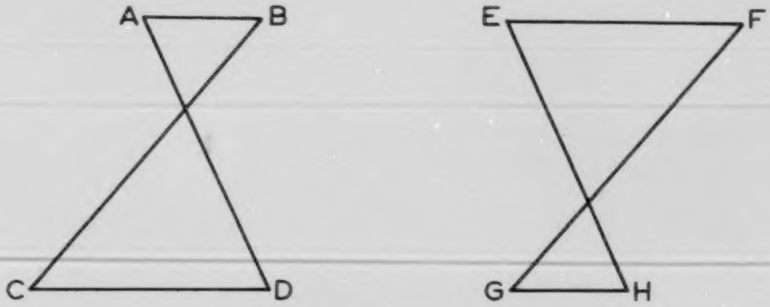
which is the first example that we gave, can be written in one line like this:

$$ABC \leftrightarrow EFD.$$

Here it should be understood that the first letter on the left corresponds to the first letter on the right, the second corresponds to the second, and the third to the third, like this:



Let us take one more example.



These two figures are of the same size and shape. To show how one can be moved onto the other, we should match up the vertices like this:

A ↔ H
B ↔ G
C ↔ F
D ↔ E.

These two figures are congruent, because the correspondence that we have written down is a congruence, that is, the figures can be made to fit if the vertices are matched in the given way. For short, we can write the congruence in one line, like this:

ABCD ↔ HGFE.

Notice that the order in which the matching pairs are written does not matter. We could have written our list of matching pairs this way:

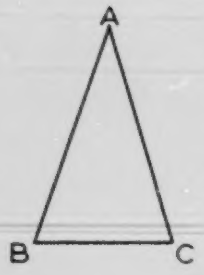
D ↔ E
B ↔ G
C ↔ F
A ↔ H;

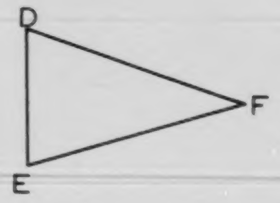
and we could have described our one-to-one correspondence in one line, like this:

DBCA ↔ EGFH.

All that matters is which point is matched with which.

It is quite possible for two figures to be congruent in more than one way.





Here the correspondence

ABC ↔ FDE

is a congruence, and the correspondence

ABC ↔ FED

is a different congruence between the same two figures.

Obviously $\triangle ABC$ coincides with itself. If we agree to match every vertex with itself, we get the congruence

ABC ↔ ABC.

This is called the identity congruence. There is another way of matching up the vertices of this triangle, however. We can use the correspondence

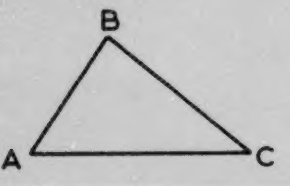
ABC ↔ ACB.

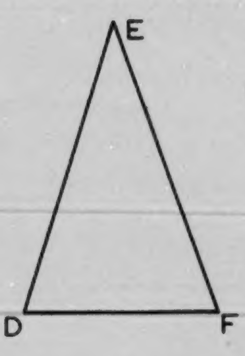
Under this correspondence, the figure is made to coincide with itself, with the vertices B and C interchanged. This is not possible for all triangles by any means; it won't work unless at least two sides of the triangle are of the same length.

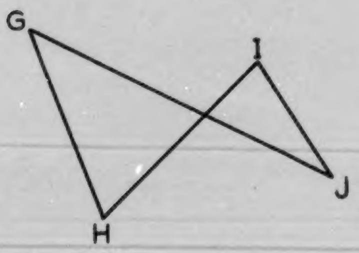
Problem Set 5-1

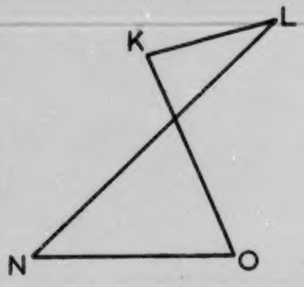
In the problems of this section, there are no tricks in the way that the figures are drawn. That is, correspondences that look like congruences when the figures are measured with reasonable care really are supposed to be congruences. In this section we are not trying to prove things. We are merely trying to learn, informally, what the idea of a congruence is all about.

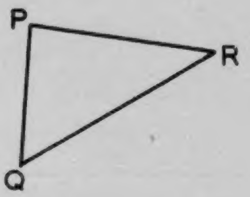
1. Below there are six figures. Write down as many congruences as you can, between these figures. (Do not count the identity congruence between a figure and itself but recall that there is a congruence between a triangle having two congruent sides and itself that is not the identity.) You should get 6 congruences in all. (One congruence is DEF ↔ SUT.)

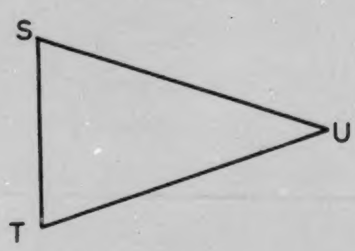




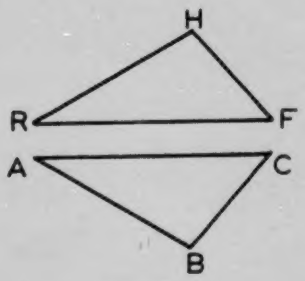


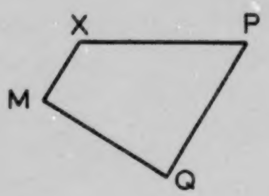


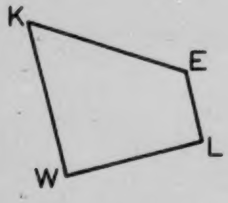


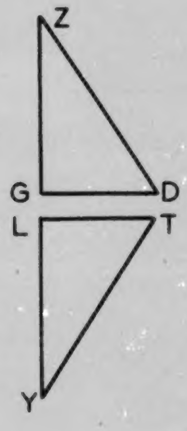


2. Answer as in Problem 1:

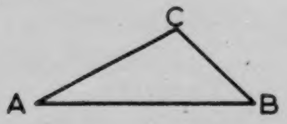


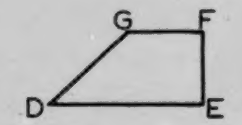


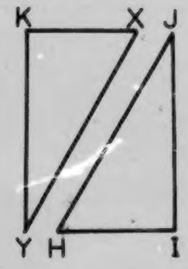


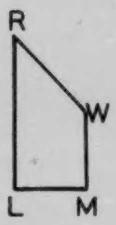


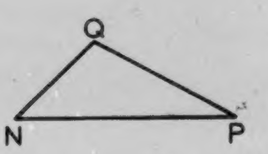
3. Answer as in Problem 1:



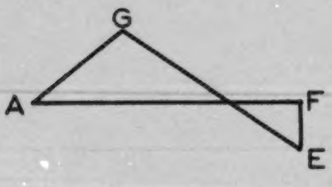


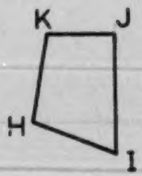


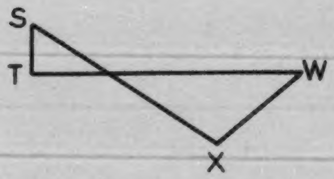


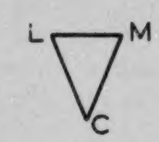


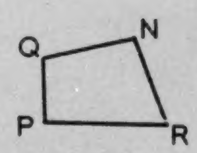
4. Answer as in Problem 1:

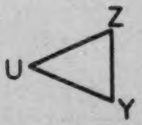




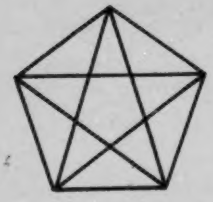




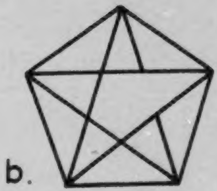




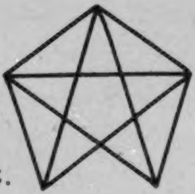
5. Name the figures that do not have a matching figure.



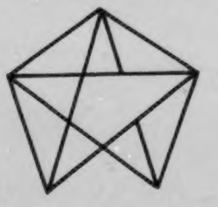
a.



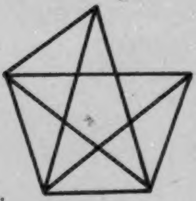
b.



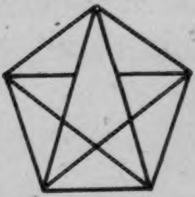
c.



d.



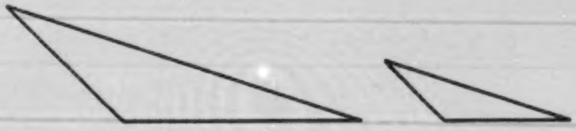
e.



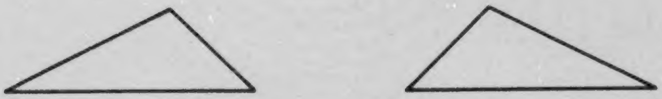
f.

6. Which pairs of the following figures are congruent?

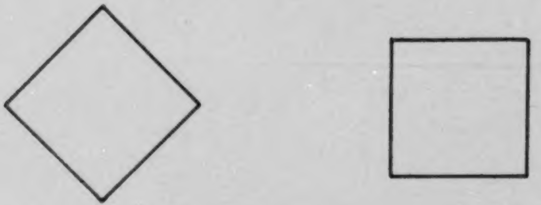
a.



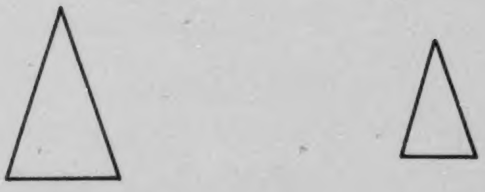
b.



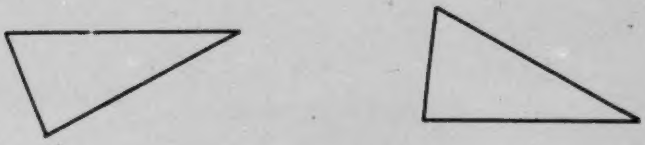
c.



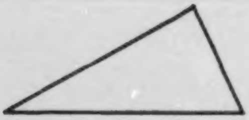
d.

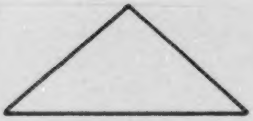


e.

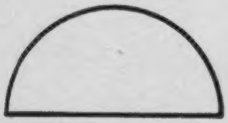


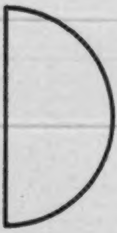
f.



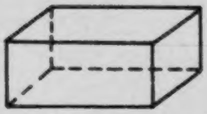


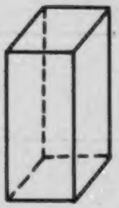
g.



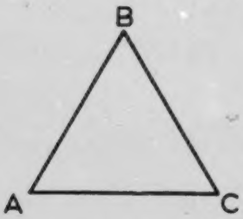


h.



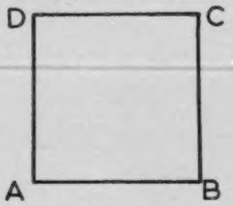


7. The triangle below is equilateral. That is, AB = AC = BC.



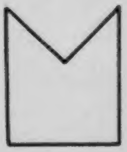
For the triangle on the preceding page, write down all congruences between the triangle and itself, starting with the identity congruence ABC ⟷ ABC. (You should get more than four congruences.)

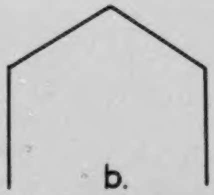
8. Write down all of the congruences between a square and itself.

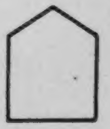


9. a. If two figures are each congruent to a third, are they congruent to each other?
   b. Is a figure congruent to itself?
   c. Can a triangle be congruent to a square?
   d. Are the top and bottom faces of a cube congruent?
   e. Are two adjacent faces of a cube congruent?
   f. Are the top and bottom faces of a rectangular block, such as a brick, congruent?
   g. Are two adjacent faces of a brick congruent?

10. Pick out the pairs of congruent figures.


a.


b.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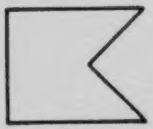
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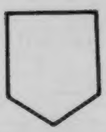
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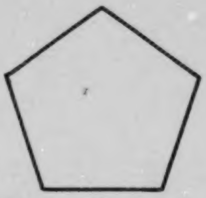
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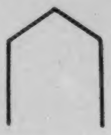
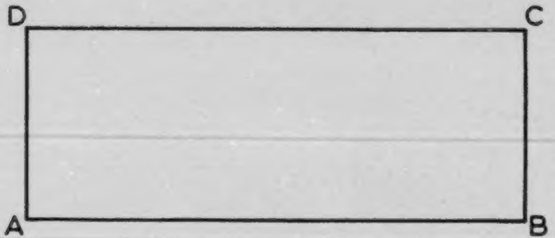
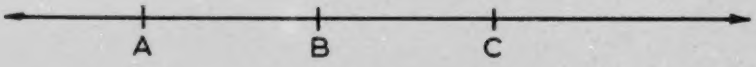

g.

11. Write down the four congruences of this figure with itself.



12. Suppose A, B, and C are three points of a line as shown with AB = BC.



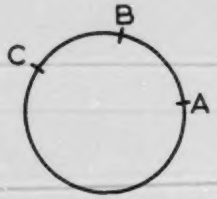
a. Describe a motion of the line that takes A to where B was. Does it necessarily take B to C?

b. Describe a motion of the line that interchanges A and C.

13. Under what conditions can the following pairs of figures be made to coincide by moving one in space without changing its size and shape? (It is understood that this moving is done abstractly in the mind. One figure can move through another so that a solid can be moved onto another solid of the same size and shape. For example, one segment can be moved to coincide with another if they have the same length. One sphere can be moved to coincide with another if their radii are the same length.)

a. Two segments.
b. Two angles.
c. Two rays.
d. Two circles.
e. Two cubes.
f. Two points.
g. Two lines.

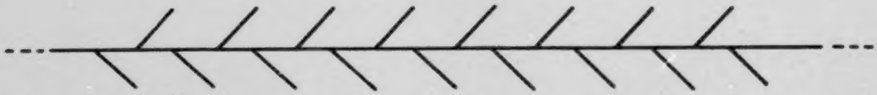
14. Given a circle containing three points A, B, C as shown, with the arc from A to B the same length as the arc from B to C.



a. Describe how the circle may be moved to take A to where B was and B to where C was.

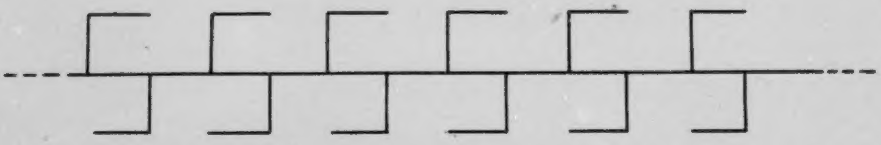
b. Describe how the circle may be moved to leave B fixed but to interchange A and C.

15. Suppose that the following ornamental frieze extends infinitely in both directions, as a line does.

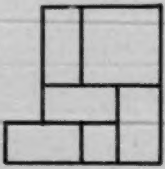


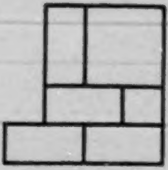
a. Describe motions of two different types that induce congruences of the frieze with itself. How many such congruences are there altogether?

b. Do the same for this frieze.



16. Which of the following figures can be fitted onto each other? For each matched pair, tell whether you must turn the figure over in space as well as slide and rotate it in a plane to make it fit on the other so that all segments fit.


a.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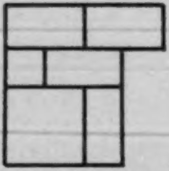
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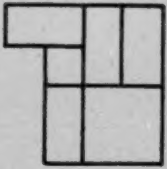
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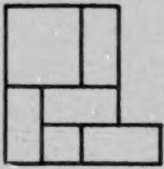
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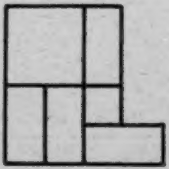
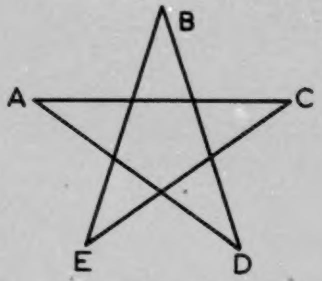

f.

17. The figure below is a five-pointed star.



Write down all of the congruences between the star and itself. To save time and paper, let us agree that a congruence for this figure is sufficiently described if we say where the points A, B, C, D, E of the star are supposed to go. For example, one of the congruences that we are looking for can be written as ABCDE $\longleftrightarrow$ BCDEA.

5-2. Congruences between Triangles.

In the preceding section, we have explained the basic idea of what a congruence is. Let us now give some mathematical definitions so that we can talk about congruence in a careful way, in terms of distance and angular measure, instead of having to talk loosely about things falling on each other.

For angles and segments, it is easy to say exactly what we mean:

Definitions. Angles are *congruent* if they have the same measure. Segments are *congruent* if they have the same length. The first definition above is merely a repetition from Section 4-3.

Analogous to Theorem 4-2 for angles we have a theorem for segments:

Theorem 5-1. Every segment is congruent to itself.

We sometimes refer to these two theorems by the term *identity congruence*.

Just as we indicate that $\angle A$ and $\angle B$ are congruent, by writing $\angle A \cong \angle B$, so we may write

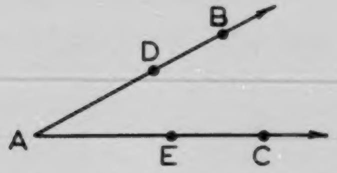
$$\overline{AB} \cong \overline{CD}$$

to indicate that the segments $\overline{AB}$ and $\overline{CD}$ are congruent. In the table below, the equation on the left and the congruence on the right in each line may be used interchangeably:

1. $m \angle A = m \angle B$.	1. $\angle A \cong \angle B$.
2. $AB = CD$.	2. $\overline{AB} \cong \overline{CD}$.

Each of the *equations* on the left is an *equation between numbers*. The first says that $m \angle A$ and $m \angle B$ are exactly the same number. The second says that the distance AB and the distance CD are exactly the same number.

Each of the *congruences* on the right is a *congruence between geometric figures*. We do not write $=$ between two geometric figures unless we mean that the figures are exactly the same, and occasions when we mean this are rare. One example is this:



Here it is correct to write

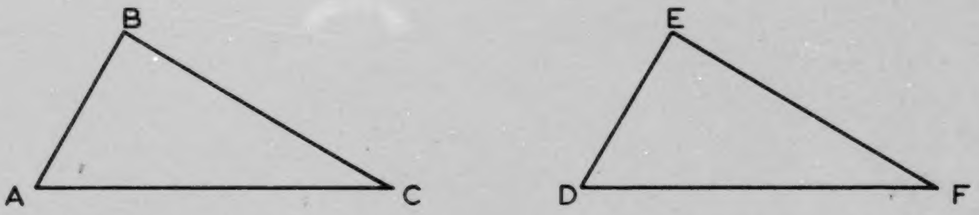
$$\angle BAC = \angle EAD,$$

because $\angle BAC$ and $\angle EAD$ are not merely congruent, they are exactly the same angle. Similarly, $\overline{AB}$ and $\overline{BA}$ are always exactly the same segment, and so it is correct to write $\overline{AB} = \overline{BA}$.

Consider now a correspondence

$$ABC \longleftrightarrow DEF$$

between the vertices of two triangles $\triangle ABC$ and $\triangle DEF$.



This automatically gives us a correspondence between the sides of the triangles, like this:

$$\overline{AB} \longleftrightarrow \overline{DE}$$
$$\overline{AC} \longleftrightarrow \overline{DF}$$
$$\overline{BC} \longleftrightarrow \overline{EF}$$

and it gives us a correspondence between the angles of the two triangles, like this:

$$\angle A \longleftrightarrow \angle D$$
$$\angle B \longleftrightarrow \angle E$$
$$\angle C \longleftrightarrow \angle F.$$

We can now state the definition of a congruence between two triangles.

Definition. Given a correspondence

$$ABC \longleftrightarrow DEF$$

between the vertices of two triangles. If every pair of corresponding sides are congruent, and every pair of corresponding angles are congruent, then the correspondence $ABC \longleftrightarrow DEF$ is a congruence between the two triangles.

You should read this definition at least twice, very carefully, to make sure that it says what a definition of the idea of a congruence between triangles ought to say.

There is a shorthand for writing congruences between triangles. When we write

$$\angle A \cong \angle D,$$

this means that the two angles $\angle A$ and $\angle D$ are congruent. (That is, $m\angle A = m\angle D$.) Similarly, when we write

$$\triangle ABC \cong \triangle DEF,$$

this means that the correspondence

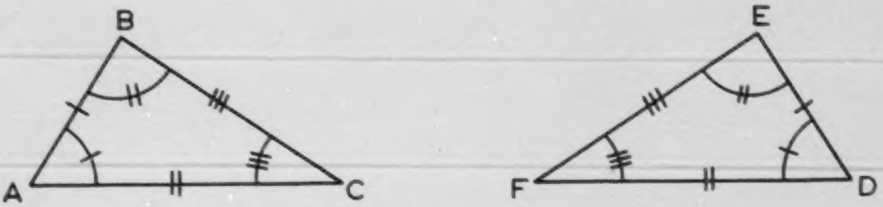
$$ABC \longleftrightarrow DEF$$

is a congruence. Notice that this is a very efficient shorthand: the single expression $\triangle ABC \cong \triangle DEF$ tells us six things at once; namely,

$AB = DE$	$\overline{AB} \cong \overline{DE}$
$AC = DF$	$\overline{AC} \cong \overline{DF}$
$BC = EF$	$\overline{BC} \cong \overline{EF}$
$m\angle A = m\angle D$	$\angle A \cong \angle D$
$m\angle B = m\angle E$	$\angle B \cong \angle E$
$m\angle C = m\angle F.$	$\angle C \cong \angle F.$

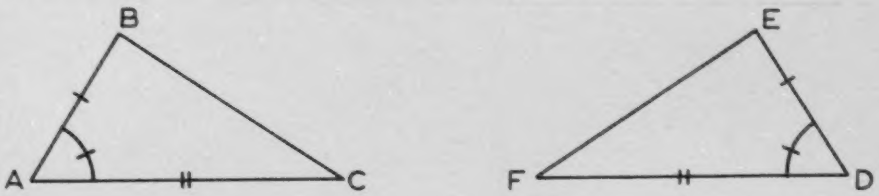
In each of these six lines, the equations on the left and the congruences on the right mean the same thing, and we can choose either notation at any time, according to convenience. Usually we will write $AB = DE$, instead of $\overline{AB} \cong \overline{DE}$, simply because it is easier to write. For the same reason, we will usually write $\angle A \cong \angle D$ instead of $m\angle A = m\angle D$.

It is sometimes convenient to indicate a congruence graphically by making marks on the corresponding sides and angles, like this:



$\triangle ABC \cong \triangle DEF$

We can also use this method to indicate that certain corresponding parts of two figures are congruent, whether or not we know about other parts.



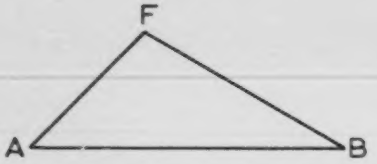
The marks in the figure indicate that (1) $AB = DE$, (2) $AC = DF$ and (3) $m\angle A = m\angle D$.

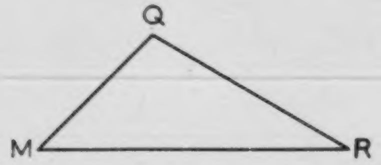
Question: Would it be correct to write $AB \cong DE$, or $\angle A = \angle D$? Why or why not?

It seems pretty clear, in the above figure, that the congruences we have indicated are enough to guarantee that the correspondence $ABC \longleftrightarrow DEF$ is a congruence. That is, if these three pairs of corresponding parts are congruent, the triangles must also be congruent. In fact, this is the content of the basic congruence postulate, to be stated in the next section.

Problem Set 5-2

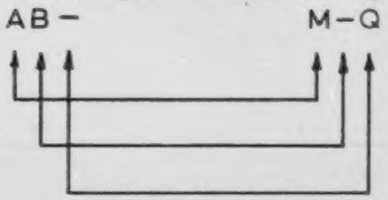
1.



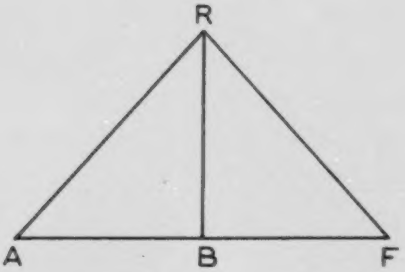


$\triangle ABF \cong \triangle MRQ$. Complete the following list by telling what should go in the blanks.

$\angle A \cong \angle M$. $\overline{AF} \cong$ ___.
$\angle B \cong$ ___. $\overline{AB} \cong$ ___.
$\angle F \cong$ ___. $\overline{FB} \cong$ ___.



2.



$\triangle ABR \cong \triangle FBR$. List the six pairs of corresponding, congruent parts of these two triangles.

3. $\triangle MRK \cong \triangle FHW$. List the six pairs of corresponding, congruent parts of these triangles. (It is not necessary to have a picture but you may make a sketch if you wish.)

4. $\triangle RQF \cong \triangle ABX$. List the six pairs of corresponding congruent parts of these triangles. Do not use a figure.

5. $\triangle AZW \cong \triangle BZW$. List the six pairs of corresponding, congruent parts of these triangles.

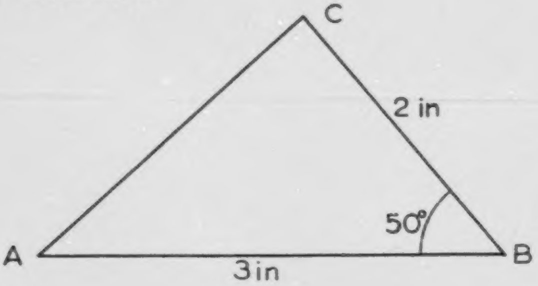
6. Here is a list of the six pairs of corresponding parts of two congruent triangles. Give the names of the two triangles that would fit in the blanks below.

$\overline{AB} \cong \overline{MK}$. $\angle A \cong \angle M$.
$\overline{BW} \cong \overline{KF}$. $\angle B \cong \angle K$.
$\overline{AW} \cong \overline{MF}$. $\angle W \cong \angle F$.

$\triangle$____ $\cong$ $\triangle$____.

7. If $\triangle ABC \cong \triangle XYZ$ and $\triangle DEF \cong \triangle XYZ$, what can be said about the relationship of $\triangle ABC$ to $\triangle DEF$? State a theorem generalizing this situation.

8.



a. Using ruler and protractor, draw a triangle ABC in which AB is 3 inches long, BC is 2 inches long and angle B is 50°. Compare your triangle with those of other members of the class.

b. Draw $\triangle ABC$ in which AC is 3 inches long, BC is 4 inches long and angle C is 70°. Compare triangles.

c. Draw $\triangle ABC$ with AB 3 inches long and BC 2 inches long. Make $\angle B$ any size that suits your fancy. Compare triangles.

d. If these three exercises suggest to you an idea concerning a congruence between two triangles, try to state or write this idea for triangles in general.

9. a. Given that $\triangle ABC$ and $\triangle DEF$ do not intersect, and that X is a point between B and C. Tell which of the symbols $=$, $\cong$ may be filled in the blanks to make the statements meaningful and possibly true.

1. $\triangle ABC$ ___ $\triangle DEF$.
2. $m\angle A$ ___ $m\angle D$.
3. $\overline{AB}$ ___ $\overline{DE}$.
4. BC ___ EF.
5. $\angle B$ ___ $\angle C$.
6. $\angle ABX$ ___ $\angle ABC$.
7. $m\angle ABX$ ___ $m\angle EDF$.

b. Which of the blanks could have been filled with either $=$ or $\cong$?

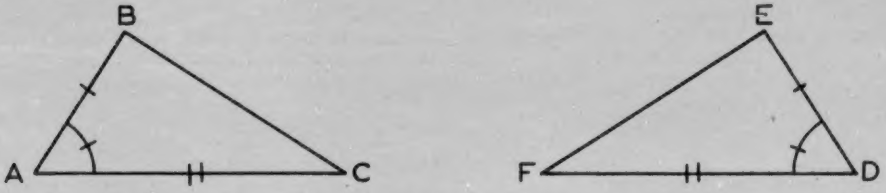
c. If $\overline{AB}$ had been the same segment as $\overline{DE}$ but if C were a different point than F, which blank could have been filled by $=$ that should otherwise have been filled by $\cong$?

5-3. The Basic Congruence Postulate.

To get at the facts on congruences of triangles, we need one new postulate. In the name of this postulate, S.A.S. stands for Side Angle Side.

Postulate 15. (The S.A.S. Postulate.) Given a correspondence between two triangles (or between a triangle and itself). If two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

To illustrate this, we repeat the previous figure.



The postulate means that if

$$\overline{AB} \cong \overline{DE},$$
$$\overline{AC} \cong \overline{DF}$$

and

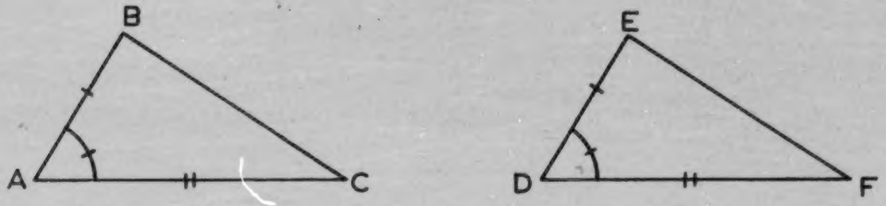
$$\angle A \cong \angle D,$$

as indicated in the figure, then

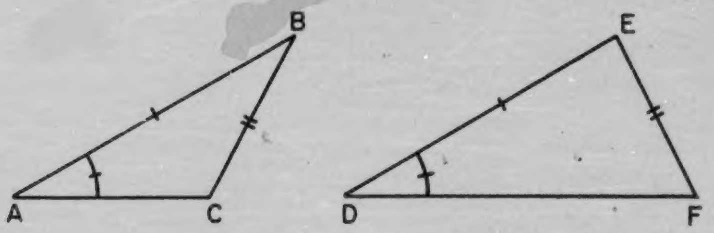
$$\triangle ABC \cong \triangle DEF;$$

that is, the correspondence ABC ⟷ DEF is a congruence.

It is very important to notice that in the S.A.S. Postulate, the given angle is the angle included between the two given sides, like this:



Under these conditions, the S.A.S. Postulate says that the correspondence ABC ⟷ DEF is a congruence. If we knew merely that some one angle and some two sides of the first triangle were congruent to the corresponding parts of the second triangle, then it would not necessarily follow that the correspondence was a congruence. For example, consider this figure:



Here AB = DE, $\angle A \cong \angle D$, BC = EF. Note that $\angle A$ and $\angle D$ are not included by the pairs of congruent sides. This correspondence is certainly not a congruence, because it matches $\overline{AC}$ with $\overline{DF}$,

$\angle B$ with $\angle E$, and $\angle C$ with $\angle F$. Since these are not congruences, the definition of congruence between triangles is not satisfied.

5-4. Writing Your Own Proofs.

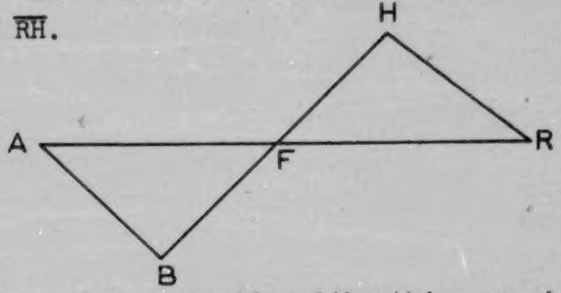
You now have enough basic material to be able to write real geometric proofs of your own. From now on, writing your own proofs will be a very important part of your work, and the chances are that it will be more fun than reading other people's proofs.

Let us take a couple of examples, to suggest how we go about finding proofs and writing them up.

Example 1. If two segments bisect each other, the segments joining the ends of the given segments are congruent.

Given: $\overline{AR}$ and $\overline{BH}$ bisect each other at F.

To prove: $\overline{AB} \cong \overline{RH}$.



Starting to work on a problem like this, we should first draw a figure and letter it, using a capital letter for each vertex. Then, state the hypothesis and conclusion in terms of the lettering of the figure.

Next, we divide the page into two columns as shown, and write in the headings Statements and Reasons.

All this, of course, isn't going to do us a bit of good unless we can think of a proof to write down.

Since our object is to prove two segments congruent, we must recall what we know about congruent segments. Looking back we can find the definition of congruent segments, of congruent triangles, and the S.A.S. Postulate. These are the available weapons about congruent segments in our arsenal, and at this point the search is short, because our arsenal is small.

To apply the postulate, we have to set up a correspondence between two triangles, in such a way that two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle. From the figure, this correspondence looks as if it ought to be

$$AFB \leftrightarrow RFH.$$

Two pairs of sides are congruent, because we have from the given data and the definition of bisect that

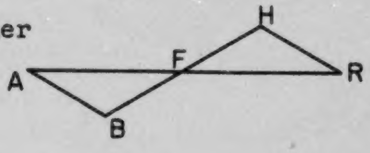
$$AF = RF \quad \text{and} \quad BF = HF.$$

How about the included angles, $\angle AFB$ and $\angle RFH$? We need to know that they are congruent, too. And they are, because they are vertical angles. Therefore, by the S.A.S. Postulate, our correspondence is a congruence. The sides $\overline{AB}$ and $\overline{RH}$ are corresponding sides, and so they are congruent. This is what we wanted to prove.

Written down in the double-column form, our proof would look like this:

Given: $\overline{AR}$ and $\overline{BH}$ bisect each other at F.

To prove: $\overline{AB} \cong \overline{RH}$.

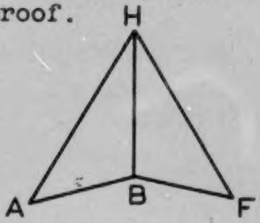


Statements	Reasons
1. $AF = RF$	1. Definition of bisect.
2. $BF = HF$.	2. Definition of bisect.
3. $\angle AFB \cong \angle RFH$.	3. Vertical angles are congruent.
4. $\triangle AFB \cong \triangle RFH$.	4. The S.A.S. Postulate.
5. $\overline{AB} \cong \overline{RH}$.	5. Definition of a congruence between triangles.

This is given merely as a sample of how your work might look. There is a limit to how "standard" we can expect the form of a proof to be. For example, in this proof we have indicated congruences between segments by writing $AF = RF$ and $BF = HF$, and so on. We could just as well have written $\overline{AF} \cong \overline{RF}$, $\overline{BF} \cong \overline{HF}$, and so on, because in each case the congruence between the segments and the equation between the distances mean the same thing.

There are only two really important things in writing proofs. First, what you write should say what you really mean. Second, the things that you really mean should form a complete logical explanation of why the theorem is true.

By now, you should have the idea, and so we give our second example in an incomplete form. Your problem is to fill in the blank spaces in such a way as to get a proof.



Example 2.

Given: $\overline{AH} \cong \overline{FH}$.

$\overrightarrow{HB}$ bisects $\angle AHF$.

To prove: $\angle A \cong \angle F$.

Statements	Reasons
1. $\overline{AH} \cong \overline{FH}$.	1. Given.
2.	2. Definition of the bisector of an angle.
3. $\overline{HB} \cong \overline{HB}$.	3. Every segment is congruent to itself.
4.	4.
5. $\angle A \cong \angle F$.	5.

A mistake often made in proofs is that the student assumes as true the very thing he is trying to prove to be true. Another common mistake is to use as a reason in his proof a theorem which is actually a consequence of the fact that he is trying to prove. Such arguments are called circular arguments, and are worthless as logical proofs.

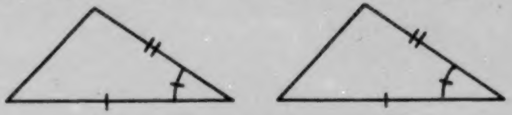
A particularly bad kind of circular argument is the use of the theorem we are trying to prove as a reason for one of the steps in its "proof".

Problem Set 5-4

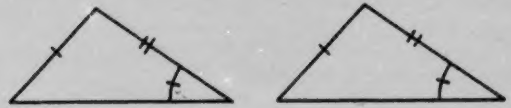
(Note: In some of the following problems we make use of a square. A square ABCD is a plane figure that is the union of four congruent segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ such that $\angle ABC$, $\angle BCD$, $\angle CDA$, $\angle DAB$ are right angles. The square will be discussed in a later chapter of the text.)

1. In each pair of triangles, if like markings indicate congruent parts, which triangles could be proved congruent by S.A.S.?

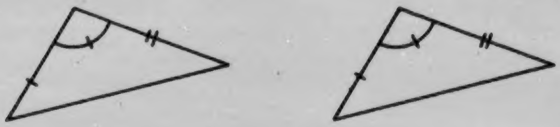
a.



b.



c.



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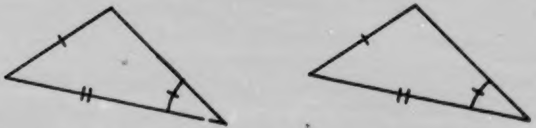

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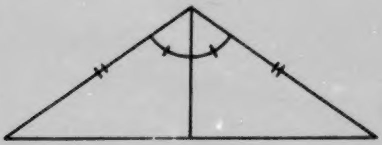

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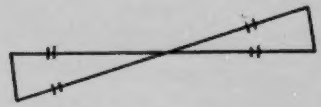

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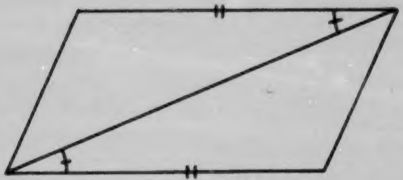

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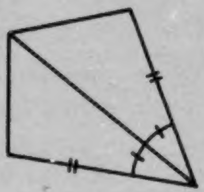
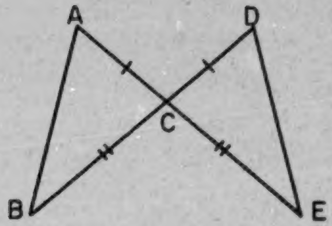
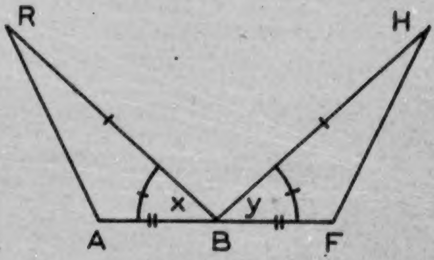


2. In the figure it is given that $\overline{AE}$ intersects $\overline{BD}$ at C, that $AC = DC$ and $BC = EC$. Show (i.e., prove) that $\angle B \cong \angle E$. Copy the following proof and supply the missing reasons.



Statements	Reasons
1. $AC = CD$.	1. Given.
2. $BC = EC$.	2. ______________.
3. $\angle ACB \cong \angle DCE$.	3. ______________ angles are congruent.
4. $\triangle ACB \cong \triangle DCE$.	4. ______________. [Note that Statement 3 refers to angles and Statement 4 to triangles, so that your reason here should refer to triangles].
5. $\angle B \cong \angle E$.	5. Corresponding parts of congruent triangles are _____.

3. Suppose in this figure $\overline{RB} \cong \overline{HB}$, $\angle x \cong \angle y$ and B is the midpoint of $\overline{AF}$. Show that $\angle R \cong \angle H$.

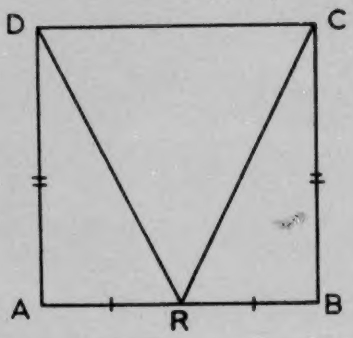


Copy and complete the following proof.

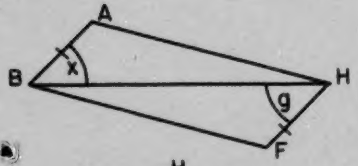
Statements	Reasons
1. $\overline{RB} \cong \overline{HB}$.	1. ______________.
2. $\angle x \cong \angle y$.	2. Given.
3. ____ = ____.	3. From the definition of midpoint.
4. ______________.	4. S.A.S.
5. ______________.	5. ______________.

4. a. If ABCD is a square and R is the midpoint of $\overline{AB}$, prove that RC = RD. (See note preceding Problem 1.)

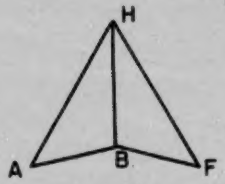
b. What pairs of congruent acute angles appear in the figure? Prove your answer.



5. In this figure AB = FH and $m\angle x = m\angle g$. Show that $m\angle A = m\angle F$.

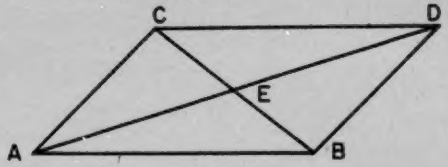


6. In this figure it is given that $m\angle ABH = m\angle FBH$, AB = FB. Prove AH = FH.



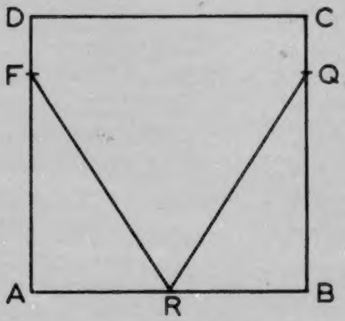
7. Prove that if segments $\overline{AH}$, $\overline{RB}$ bisect each other at point F, then $\triangle FAB \cong \triangle FHR$.

8. Prove: If the line segments $\overline{AD}$ and $\overline{BC}$ bisect each other, then AB = DC and AC = DB.

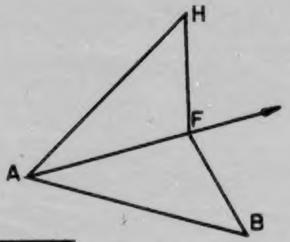


9. a. Given: Square ABCD, R is the midpoint of $\overline{AB}$, F is a point between A and D, Q is a point between C and B, DF = CQ. To prove: RF = RQ.

b. Are there two other points F', Q' of square ABCD not on $\overline{AD}$ or $\overline{BC}$ such that RF' = RQ'? Where are they?

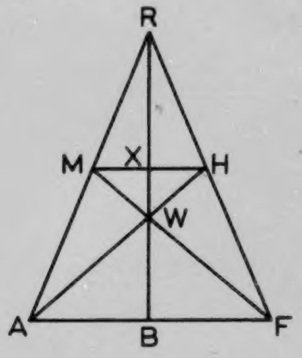


10. Suppose in this figure that AB = AH and that $\overrightarrow{AF}$ bisects $\angle HAB$. Prove that FH = FB.



5-5. Overlapping Triangles. Using the Figure in Statements.

Frequently in geometric figures, the triangles that we need to work with are not entirely separate but overlap, like $\triangle AFM$ and $\triangle FAH$ in the figure below.



The easiest way to avoid getting mixed up, and making mistakes, in dealing with such cases, is to write down congruences in a standard form, like this,

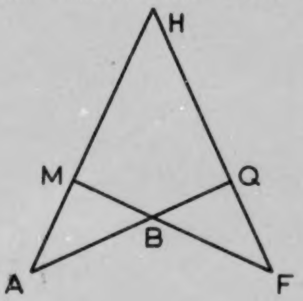
$$\triangle AFM \cong \triangle FAH.$$

Check that the correspondence $AFM \leftrightarrow FAH$ really is a congruence, and then later refer back to $\triangle AFM \cong \triangle FAH$ when we want to conclude that two corresponding sides (or corresponding angles) are congruent.

Of course, if you don't see the congruences between the overlapping triangles, you will have nothing to check and nothing to apply later. To practice up, write all the congruences that you can between triangles contained in the figure above, if it is given that $AR = FR$ and M, H, B are the midpoints of the respective sides.

Let us now look at a case in which this sort of thing comes up in the proof of a theorem.

Given: $HA = HF$.
$HM = HQ$.
To prove: $FM = AQ$.



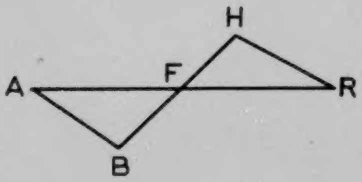
A very common way to prove that two segments are congruent is to show that the segments are corresponding sides of congruent triangles. If this way can be used successfully here, then the first thing to do is locate the triangles which contain $\overline{FM}$ and $\overline{AQ}$. These are $\triangle HMF$ and $\triangle HQA$, and these triangles overlap quite a bit. Now the problem becomes one of proving the triangles congruent. The proof in the double-column form goes like this:

Statements	Reasons
1. $HA = HF$.	1. Given.
2. $\angle H \cong \angle H$.	2. An angle is congruent to itself.
3. $HM = HQ$.	3. Why?
4. $\triangle HMF \cong \triangle HQA$.	4. Why?
5. $FM = AQ$.	5. Why?

A strictly logical proof must not depend on a figure but must follow from the postulates, the definitions, and the previously proved theorems. But geometers in practice use figures as a matter of convenience, and readily accept many observable facts without a tedious restatement in words, unless such a restatement is essential to clarifying the problem at hand.

To illustrate, let us look at a restatement of Example 1 used previously.

Example 1. Let A, B, F, H and R be five non-collinear points lying in a plane. If (1) F is between A and R, (2) F is between B and H, (3) $AF = FR$, and (4) $BF = FH$, then (5) $AB = RH$.
This conveys all the information conveyed by the figure on the left and the notation on the right below.



Given: $\overline{AR}$ and $\overline{BH}$ bisect each other at F.

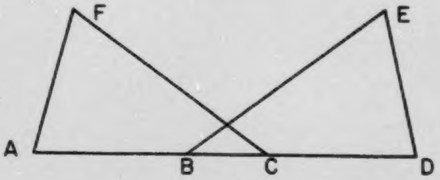
To prove: $\overline{AB} \cong \overline{RH}$.

Notice that (1) tells us that $\overrightarrow{FA}$ and $\overrightarrow{FR}$ are opposite rays, and (2) tells us that $\overrightarrow{FB}$ and $\overrightarrow{FH}$ are opposite rays. These two things, taken together, mean that $\angle AFB$ and $\angle RFH$ are vertical angles. (See definition of vertical angles.) This is the sort of information that we normally read from a figure.

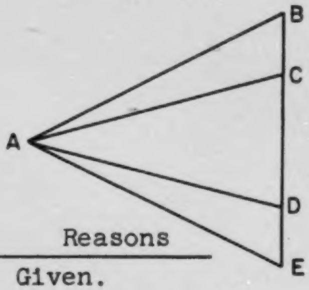
In stating problems in this text we will frequently avoid tedious repetition by referring to a figure. You can use the figure to give the collinearity of points, the order of points on a line, the location of a point in the interior or exterior of an angle or in a certain half-plane, and, in general, the relative position of points, lines, and planes. Things you cannot assume because "they look that way" to you are the congruence of segments or angles, that a certain point is a midpoint of a segment, that two lines are perpendicular, nor that two angles are complementary.

Problem Set 5-5

1. If in this figure
   $AC = DB$,
   $\angle ACF \cong \angle DBE$ and
   $FC = EB$,
   prove that $AF = DE$.

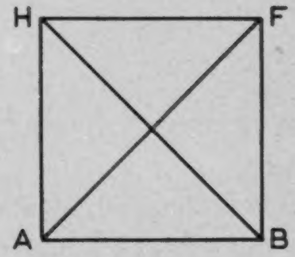


2. In this figure $BC = ED$
   $AC = AD$ and
   $\angle ACE \cong \angle ADB$.
   Prove $\triangle ACE \cong \triangle ADB$.

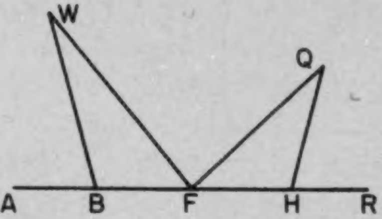


Proof: (Fill in the blanks.)

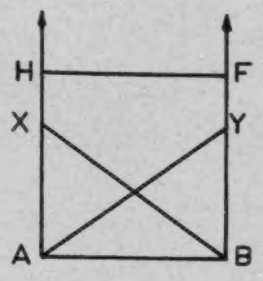
Statements	Reasons
1. $BC = ED$.	1. Given.
2. $CD = DC$.	2. ____________.
3. $BD = EC$.	3. Addition, from statements 1 and 2.
4. $AC = AD$.	4. ____________.
5. $\angle ACE \cong \angle ADB$.	5. ____________.
6. ______________.	6. ____________.



3. Prove that the diagonals of a square are of equal length. (See note preceding Problem 1 of Problem Set 5-4.)
Given: ABFH is a square.
To prove: AF = BH.

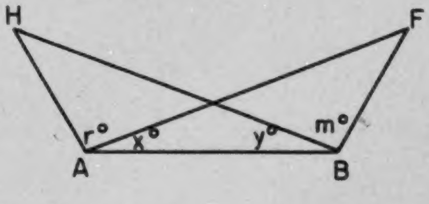


4. In this figure $\angle ABW \cong \angle RHQ$ and F is the midpoint of $\overline{BH}$. Can you prove $\triangle WBF \cong \triangle QHF$? Explain.

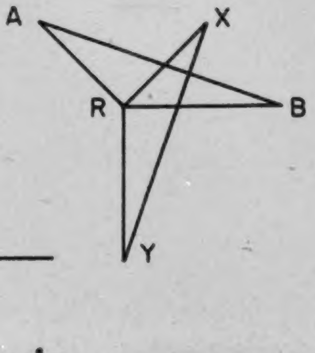


5. a. If ABFH is a square and $\overline{AX}$, $\overline{BY}$ are congruent segments on the rays $\overrightarrow{AH}$, $\overrightarrow{BF}$ respectively, show that $\overline{AY}$, $\overline{BX}$ are congruent.
Restatement:
Given: ABFH is a square.
X, Y are points of $\overrightarrow{AH}$, $\overrightarrow{BF}$, respectively.
$\overline{AX} \cong \overline{BY}$.
To prove: $\overline{AY} \cong \overline{BX}$.

b. In the figure, X is between A and H, and Y is between B and F. Would the proof be affected if H were between A and X, and F were between B and Y?



6. Suppose it is given in this figure that AH = BF, r = m and x = y. Prove that HB = FA.



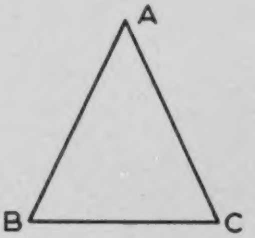
7. If in the figure $\overline{AR} \perp \overline{RX}$, $\overline{BR} \perp \overline{RY}$, AR = RX and BR = RY, prove that AB = XY.

5-6. The Isosceles Triangle Theorem. The Angle Bisector Theorem.

At the end of Section 5-1 we mentioned the case of matching up the vertices of a triangle $\triangle ABC$ in which at least two sides of the triangle are of the same length. This, in fact, is the case that we deal with in our first formally stated congruence theorem:

Theorem 5-2. If two sides of a triangle are congruent, then the angles opposite these sides are congruent.

Restatement: Given a triangle $\triangle ABC$. If $AB = AC$, then $\angle B \cong \angle C$.



Proof: Consider the correspondence

$$ABC \longleftrightarrow ACB,$$

between $\triangle ABC$ and itself. Under this correspondence, we see that

$$\overline{AB} \longleftrightarrow \overline{AC},$$
$$\overline{AC} \longleftrightarrow \overline{AB},$$
$$\angle A \longleftrightarrow \angle A.$$

Thus two sides and the included angle of $\triangle ABC$ are congruent to the parts that correspond to them. By the S.A.S. Postulate, this means that

$$\triangle ABC \cong \triangle ACB,$$

that is, the correspondence $ABC \longleftrightarrow ACB$ is a congruence. By the definition of a congruence between triangles all pairs of corresponding parts are congruent. Therefore

$$\angle B \cong \angle C,$$

because these angles are corresponding parts.

We now show how the above proof looks in two-column form. The same figure is used.

Theorem 5-2. If two sides of a triangle are congruent, then the angles opposite these sides are congruent.

Given: $\triangle ABC$ with $\overline{AB} \cong \overline{AC}$.

To prove: $\angle B \cong \angle C$.

Proof:

Statements	Reasons
1. $\overline{AB} \cong \overline{AC}$. $\overline{AC} \cong \overline{AB}$.	1. Given
2. $\angle A \cong \angle A$.	2. Identity congruence.
3. $\triangle ABC \cong \triangle ACB$.	3. Steps 1 and 2 and the S.A.S. Postulate.
4. $\angle B \cong \angle C$.	4. Definition of a congruence between triangles.

Usually, we will state theorems in words, as we have stated Theorem 5-2, and then restate them, using notation which will be the notation of the proof.

Definitions. A triangle with two congruent sides is called isosceles. The remaining side is the base. The two angles that include the base are base angles.

In these terms, we can state Theorem 5-2 in this form: "The base angles of an isosceles triangle are congruent."

Definitions. A triangle whose three sides are congruent is called equilateral. A triangle no two of whose sides are congruent is called scalene.

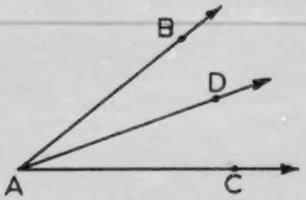
Definition. A triangle is equiangular if all three of its angles are congruent.

Using the term equiangular we state a theorem which readily follows from Theorem 5-2. We denote this theorem as Corollary 5-2-1. A corollary is a theorem which is an easy consequence of another theorem. The proof of Corollary 5-2-1 is left for you to do.

Corollary 5-2-1. Every equilateral triangle is equiangular,

In proving theorems for yourself, you will need to make your own figures. It is important to draw figures in such a way that they remind you of what you know, without suggesting more than you know. For example, the figure given in the proof of Theorem 5-2 looks like an isosceles triangle, and this is as it should be, because the hypothesis of the theorem says that the triangle has two congruent sides. In the figure for the S.A.S. Postulate, it looks as if $\triangle ABC \cong \triangle DEF$, and this is as it should be, because this is the situation dealt with in the postulate. But it would not have been good to draw isosceles triangles to illustrate the S.A.S. Postulate, because this would suggest things that the postulate doesn't say.

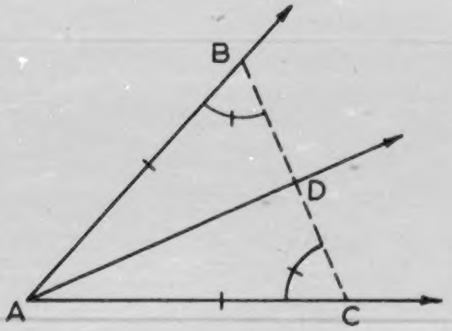
Definition. A ray $\overrightarrow{AD}$ bisects, or is a bisector of, an angle $\angle BAC$ if D is in the interior of $\angle BAC$, and $\angle BAD \cong \angle DAC$.



Note that if $\overrightarrow{AD}$ bisects $\angle BAC$, then $m\ \angle BAD = m\ \angle DAC = \frac{1}{2}\ m\ \angle BAC$.

Theorem 5-3. Every angle has exactly one bisector.

Proof: Given $\angle A$. By the Point Plotting Theorem we can find B and C, points on the sides of $\angle A$, such that (1) $AB = AC$.



Let D be the mid-point of BC, so that (2) $DB = DC$. Since $AB = AC$, it follows by Theorem 5-2 that (3) $\angle B \cong \angle C$. (This follows even though the isosceles triangle $\triangle ABC$ is "lying on its side.") From (1), (2) and (3), and the S.A.S. Postulate it follows that

$$\triangle ABD \cong \triangle ACD.$$

Therefore, $\angle BAD \cong \angle CAD$, and so $m\angle BAD = m\angle CAD$. By the definition of bisector of an angle, this means that $\overrightarrow{AD}$ bisects $\angle BAC$.

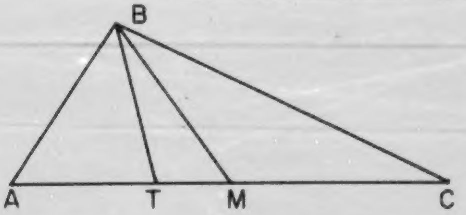
To justify our use of the word "exactly" we must prove that $\overrightarrow{AD}$ is the only ray having this property. Suppose there is a ray $\overrightarrow{AE}$ also a bisector of $\angle A$. Then $m\angle CAD = m\angle CAE$, since each of these equals $\frac{1}{2}m\angle BAC$. Applying the Angle Construction Postulate to the half-plane with $\overleftrightarrow{AC}$ as edge shows that we must have $\overrightarrow{AE} = \overrightarrow{AD}$, that is, $\overrightarrow{AE}$ and $\overrightarrow{AD}$ stand for the same ray. Hence, there is exactly one bisector.

The following definitions are useful in discussing properties of triangles.

Definition. A median of a triangle is a segment whose end-points are one vertex of the triangle and the mid-point of the opposite side.

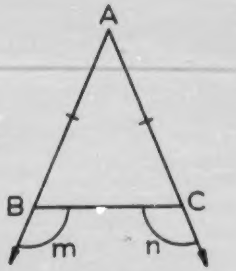
Definition. An angle bisector of a triangle is a segment whose end-points are one vertex of the triangle and a point of the opposite side which lies in the ray bisecting the angle at the given vertex.

Note that every triangle has three medians and three angle bisectors. The figure shows one median and one angle bisector of $\triangle ABC$. $\overline{BM}$ is the median from B, and $\overline{BT}$ is the angle bisector from B.



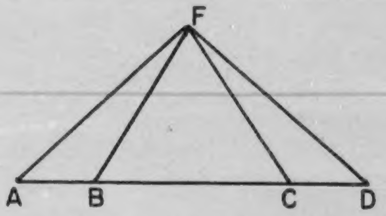
Problem Set 5-6

1. In the figure $AB = AC$. We start the proof that $\angle m \cong \angle n$. Complete this proof supplying reasons.

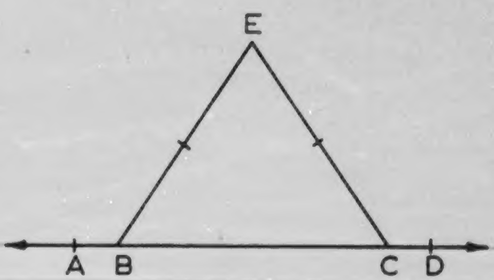


Proof: Statements	Reasons
1. $\angle ABC \cong \angle ACB$.	1. ______________.
2. $\angle m$ is supplementary to $\angle ABC$. $\angle n$ is supplementary to $\angle ACB$.	2. ______________.
3. $\angle m \cong \angle n$.	3. ______________.

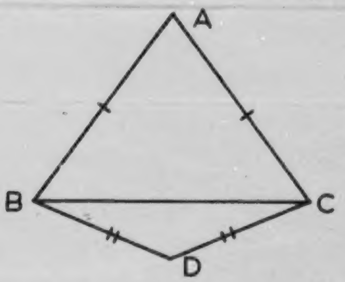
2. Given: In the figure FA = FD and AB = DC.
Prove: $\triangle AFB \cong \triangle DFC$,
$\triangle FBC \cong \triangle FCB$.



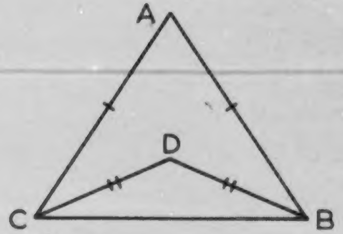
3. If in the figure $\overline{EB} \cong \overline{EC}$, prove that $\angle EBA \cong \angle ECD$.



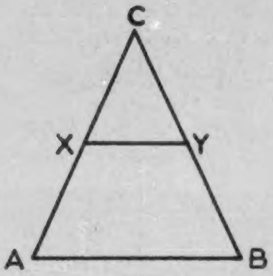
4. If AB = AC and DB = DC in the plane figure, show that $\angle ABD \cong \angle ACD$.



5. If AC = AB and CD = BD in the plane figure, show $\angle ACD \cong \angle ABD$.

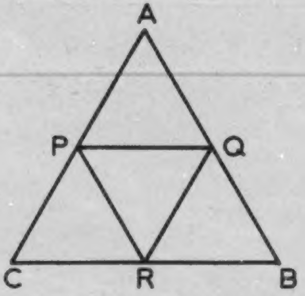


6. Give a paragraph proof rather than a two-column proof of the following:
Given: X and Y are the midpoints of the congruent sides $\overline{AC}$ and $\overline{BC}$ of the isosceles triangle ABC.
To prove: $\angle CXY \cong \angle CYX$.



7. Prove Corollary 5-2-1. (Every equilateral triangle is equiangular.)

8. Given equilateral triangle ABC with Q, R and P, the midpoints of the sides as shown. Prove that $\triangle PQR$ is equilateral.



9. Prove the following: If median $\overline{FQ}$ of $\triangle FAB$ is perpendicular to side $\overline{AB}$, then $\triangle FAB$ is isosceles.

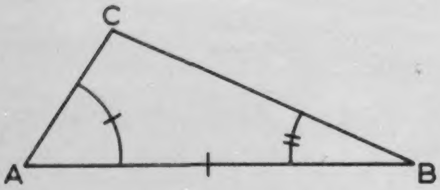
5-7. The Angle Side Angle Theorem.

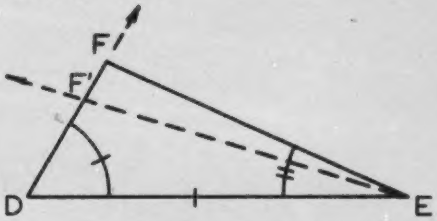
Theorem 5-4. (The A.S.A. Theorem.) Given a correspondence between two triangles, (or between a triangle and itself). If two angles and the included side of the first triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

Restatement: Let $ABC \leftrightarrow DEF$ be a correspondence between two triangles. If

$$\angle A \cong \angle D,$$
$$AB = DE,$$
$$\angle B \cong \angle E,$$

then $\triangle ABC \cong \triangle DEF$.





Proof: Statements	Reasons
1. On the ray $\overrightarrow{DF}$ there is a point F' such that $DF' = AC$.	1. The Point Plotting Theorem.
2. $AB = DE$ and $m\angle A = m\angle D$.	2. Given.
3. $\triangle ABC \cong \triangle DEF'$.	3. The S.A.S. Postulate.
4. $\angle ABC \cong \angle DEF'$.	4. Definition of a congruence between triangles.
5. $\angle ABC \cong \angle DEF$.	5. Given.
6. $\angle DEF' \cong \angle DEF$.	6. Steps 4 and 5, and the definition of congruent angles.
7. $\overrightarrow{EF}$ and $\overrightarrow{EF'}$ are the same ray.	7. Step 6 and Postulate 12.
8. $F = F'$.	8. Two lines ($\overleftrightarrow{EF}$ and $\overleftrightarrow{DF}$) intersect in at most one point.
9. $\triangle ABC \cong \triangle DEF$.	9. Statements 3 and 8.

The proofs of the following theorem and corollary are left to the student. The proofs are analogous to those of Theorem 5-2 and Corollary 5-2-1.

Theorem 5-5. If two angles of a triangle are congruent, the sides opposite these angles are congruent.

Corollary 5-5-1. An equiangular triangle is equilateral.

Problem Set 5-7

1. In some parts of this exercise there is not enough information to enable you to prove the two triangles are congruent even if you use all other facts that you know, for example, that "vertical angles are congruent". If it can be proved that the two triangles are congruent, name the statement (A.S.A. or S.A.S.) supporting your conclusion; if there is not enough information given to prove the triangles are congruent, name another pair of congruent parts that would enable you to prove them congruent. If there are two possibilities, name both.

a. Given only that $\overline{AH} \cong \overline{AB}$.

b. Given only that $\angle c \cong \angle d$.

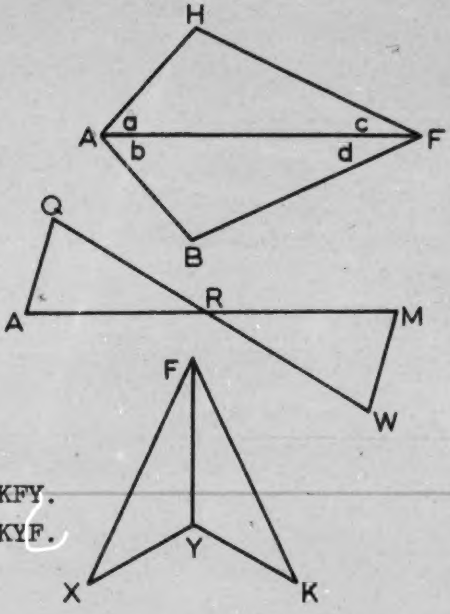
c. Given only that $\angle a \cong \angle b$ and $\angle c \cong \angle d$.

d. Given only that $\overline{AR} \cong \overline{MR}$.

e. Given only that $\angle A \cong \angle M$.

f. Given only that $\angle XFY \cong \angle KFY$.

g. Given only that $\angle XYF \cong \angle KYF$.



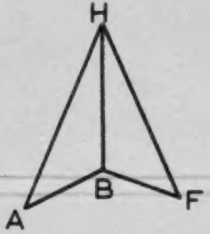
2. In accordance with the specifications at the left, list the data which would correctly fill the blanks.

a. Side, angle, side of $\triangle ABH$:
$\overline{AH}$, ___, $\overline{HB}$.

b. Angle, side, angle of $\triangle ABH$:
___, $\overline{HB}$, ___.

c. Angle, side, angle of $\triangle BFH$:
$\angle F$, ___, $\angle HBF$.

d. Side, angle, side of $\triangle BFH$:
$\overline{BF}$, ___, ___.



3. Follow the directions of Problem 2.

a. Angle, side, angle of $\triangle ABF$:
___, $\overline{BF}$, ___.

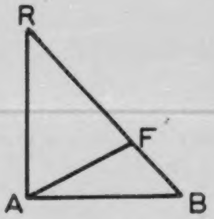
b. Side, angle, side of $\triangle RAF$:
___, $\angle R$, ___.

c. Side, angle, side of $\triangle RAB$:
___, $\angle B$, ___.

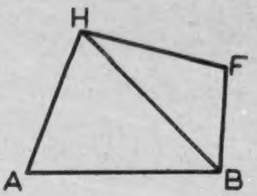
d. Side, angle, side of $\triangle RAB$:
$\overline{BR}$, ___, $\overline{RA}$.

e. Angle, side, angle of $\triangle RAF$:
$\angle R$, ___, $\angle RFA$.

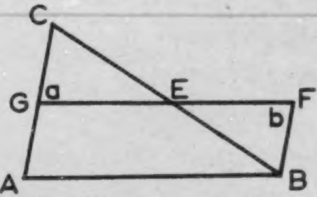
f. Angle, side, angle of $\triangle AFB$:
$\angle FAB$, $\overline{AF}$, ___.



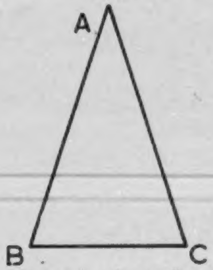
4. Follow the directions of Problem 2.
   a. Side, angle, side of $\triangle HFB$:
   ___, $\angle HBF$, ___.
   b. Angle, side, angle of $\triangle ABH$:
   ___, $\overline{HB}$, ___.
   c. Side, angle, side of $\triangle HFB$:
   $\overline{HB}$, ___, $\overline{BF}$.
   d. Angle, side, angle of $\triangle HFB$:
   ___, $\overline{BF}$, ___.
   e. Side, angle, side of $\triangle ABH$:
   $\overline{AH}$, ___, $\overline{AB}$.



5. If $\overline{CB}$ bisects $\overline{GF}$ and $\angle a \cong \angle b$ in the figure, prove that $\overline{GF}$ bisects $\overline{CB}$.



6. Prove Theorem 5-5.(If two angles of a triangle are congruent, the sides opposite these angles are congruent.)
   Restatement: If in $\triangle ABC$, $\angle B \cong \angle C$, then $AB = AC$.
   Hint: Use congruency of the triangle with itself.

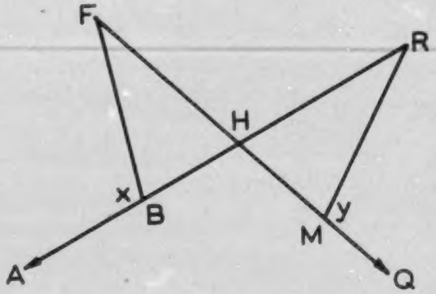


7. Prove Corollary 5-5-1. (Every equiangular triangle is equilateral.) Use a paragraph proof.

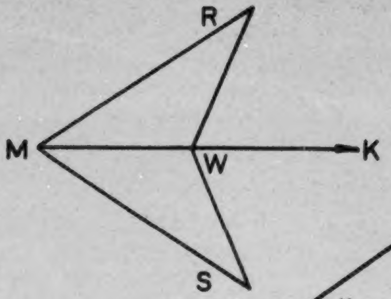
8. If $\triangle ABC$ is equilateral, prove $\triangle ABC \cong \triangle CAB$.

9. If the bisector of $\angle G$ in $\triangle FGH$ is perpendicular to the opposite side at K, then triangle FGH is isosceles.

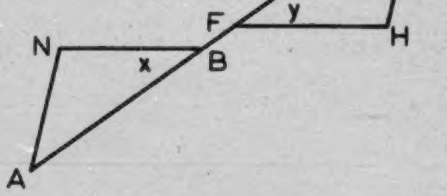
10. Given: The figure with
    $\angle x \cong \angle y$ and
    $\overline{HB} \cong \overline{HM}$.
    Prove: $\overline{HF} \cong \overline{HR}$.



11. In the figure, $\overrightarrow{MK}$ bisects $\angle RMS$ and $\angle RWK \cong \angle SWK$. Can it be proved that $\angle R \cong \angle S$? If so, do so.

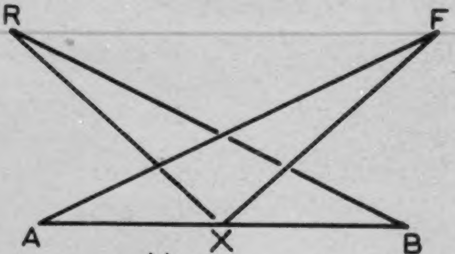


12. Prove that $\overline{AN} \cong \overline{RH}$ if $\overline{AF} \cong \overline{RB}$, $\angle A \cong \angle R$ and $\angle x \cong \angle y$ in the figure.

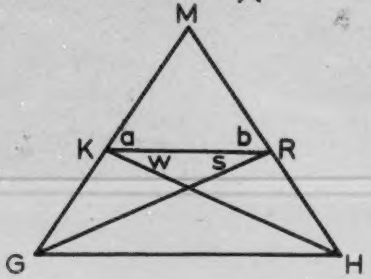


*13. a. If, in the figure, X is the midpoint of $\overline{AB}$, $\angle A \cong \angle B$ and $\angle AXR \cong \angle BXF$, show that $\overline{AF} \cong \overline{BR}$.

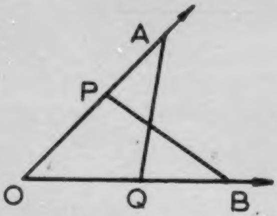
b. Do you need as a part of the hypothesis that the figure lies in a plane?



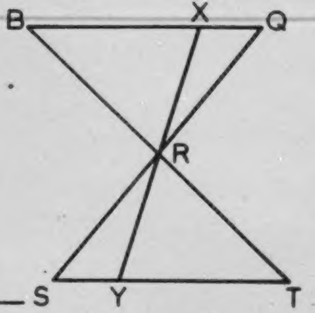
14. Given: $\angle a \cong \angle b$ and $\angle w \cong \angle s$ in the figure.
Prove: $\overline{GR} \cong \overline{KH}$.



*15. Can the following be proved on the basis of the information given?
Given: $\angle AOB$ with $OA = OB$ and P, Q, points on rays $\overrightarrow{OA}$, $\overrightarrow{OB}$ with $AQ = BP$.
Prove: $OP = OQ$.



*16. Prove that $RX = RY$ if it is given that in the figure: $BQ = TS$, $m\ \angle B = m\ \angle T$ and $m\ \angle Q = m\ \angle S$.



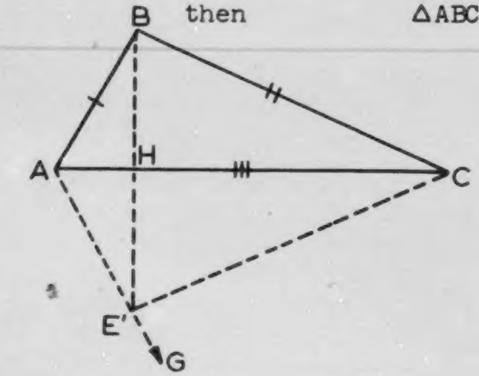
5-8. The Side Side Side Theorem.

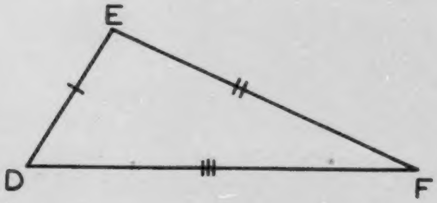
Theorem 5-6. (The S.S.S. Theorem.) Given a correspondence between two triangles (or between a triangle and itself). If all three pairs of corresponding sides are congruent, then the correspondence is a congruence.

Restatement: Let $ABC \leftrightarrow DEF$ be a correspondence between two triangles. If

$$AB = DE,$$
$$AC = DF,$$
$$BC = EF,$$

then $\triangle ABC \cong \triangle DEF$.





Proof: Statements	Reasons
1. There is a ray $\overrightarrow{AG}$ such that $\angle CAG \cong \angle FDE$, and such that B and G are on opposite sides of $\overleftrightarrow{AC}$.	1. The Angle Construction Postulate.
2. There is a point E' on $\overrightarrow{AG}$ such that $AE' = DE$.	2. The Point Plotting Theorem.
3. $\triangle AE'C \cong \triangle DEF$.	3. The S.A.S. Postulate.

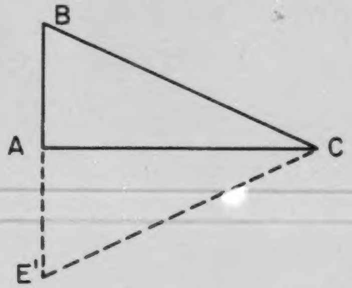
What we have done, so far, is to duplicate $\triangle DEF$ on the under side of $\triangle ABC$, using the S.A.S. Postulate.

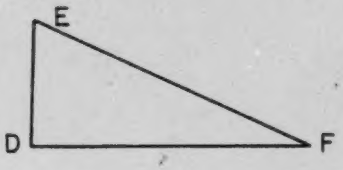
4. $AB = AE'$.	4. $AB = DE$ by hypothesis; and $DE = AE'$, from Statement 2.
5. $BC = E'C$.	5. $BC = EF$, by hypothesis; and $EF = E'C$ from Statement 3.
6. The segment $\overline{BE'}$ intersects the line $\overline{AC}$ in a point H.	6. By Statement 1, B and E' are on opposite sides of the line $\overleftrightarrow{AC}$.

We shall now complete the proof for the case in which H is between A and C, as in the figure. The other possible cases will be discussed later.

7.	$\angle ABH \cong \angle AE'H$.	7.	Statement 4 and Theorem 5-2.
8.	$\angle CBH \cong \angle CE'H$.	8.	Statement 5 and Theorem 5-2.
9.	$m\angle ABH + m\angle CBH = m\angle ABC$.	9.	The Angle Addition Postulate.
10.	$m\angle AE'H + m\angle CE'H = m\angle AE'C$.	10.	The Angle Addition Postulate.
11.	$\angle ABC \cong \angle AE'C$.	11.	Statements 7, 8, 9 and 10.
12.	$\angle ABC \cong \angle DEF$.	12.	Statements 3 and 11.
13.	$\triangle ABC \cong \triangle DEF$.	13.	Statement 12, the hypothesis, and the S.A.S. Postulate.

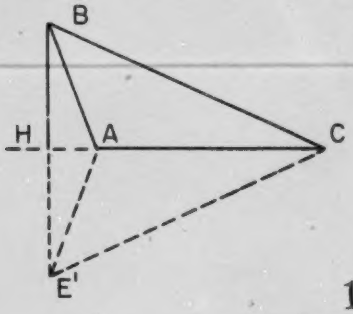
This completes the proof for the case in which H is between A and C. We recall that H is the point in which the line $\overleftrightarrow{BE'}$ intersects the line $\overleftrightarrow{AC}$. If $H = A$, then B, A and E' are collinear, and the figure looks like this:

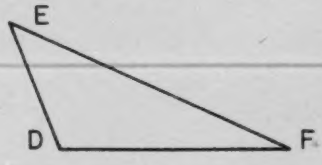




In this case $\angle B \cong \angle E'$, because the base angles of an isosceles triangle are congruent. Therefore $\angle B \cong \angle E$, because $\angle E \cong \angle E'$. The S.A.S. Postulate applies, as before, to show that $\triangle ABC \cong \triangle DEF$.

If A is between H and C, then the figure looks like this:





and we show that $\angle ABC \cong \angle E$ by subtracting the measures of angles, instead of by adding them. That is,

$$m\angle ABC = m\angle HBC - m\angle HBA$$

and $$m\angle AE'C = m\angle HE'C - m\angle HE'A,$$

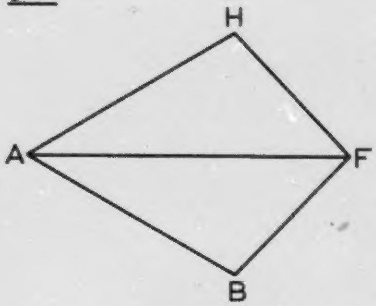
so that $$\angle ABC \cong \angle AE'C \cong \angle DEF,$$

as before. The rest of the proof is exactly the same as in the first case.

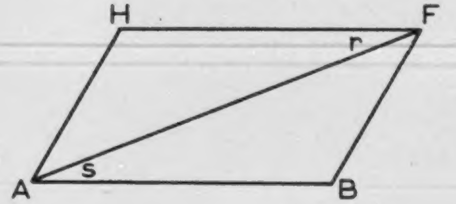
The two remaining cases, $H = C$ and C between A and H, are similar to the two above.

Problem Set 5-8

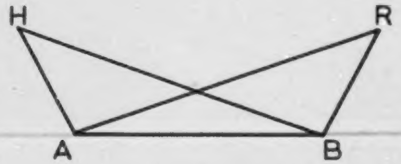
1. Given: $\triangle ABF$ and $\triangle AHF$ with $\overline{AH} \cong \overline{AB}$ and $\overline{HF} \cong \overline{BF}$.
   Prove: $\angle HAF \cong \angle BAF$.



2. In the figure, $\overline{AB} \cong \overline{FH}$ and $\overline{AH} \cong \overline{FB}$. Show that $\angle r \cong \angle s$.

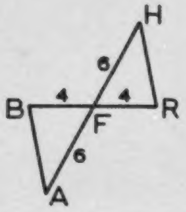


3. In the figure, $\overline{AH} \cong \overline{BR}$ and $\overline{BH} \cong \overline{AR}$. Prove that $\angle H \cong \angle R$.

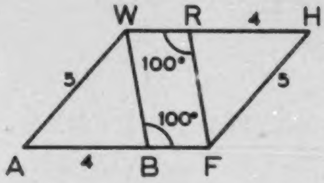


4. Consider the pairs of triangles pictured below. If on the basis of our information to date they can be proved congruent, tell which congruency statement you would use.

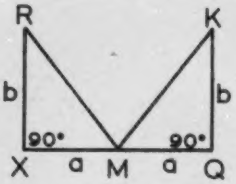
a.



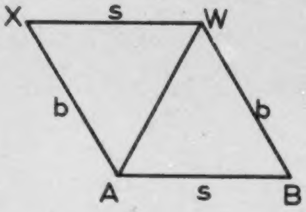
b.


W
R
4
H
5
100°
5
100°
A
4
B
F

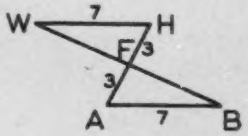
c.


R
K
b
b
90°
90°
X
a
M
a
Q

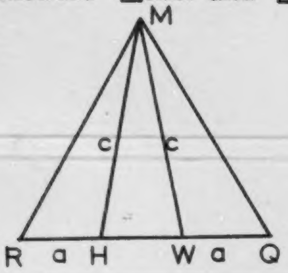
d.



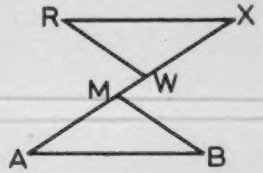
e.



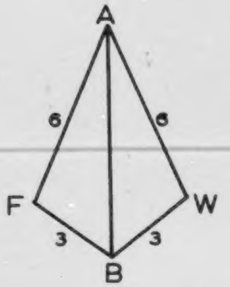
f. Consider $\triangle RWM$ and $\triangle QHM$.



g. $AW = XM$, $AB = XR$, $\angle A \cong \angle X$.

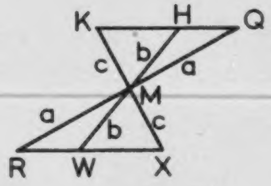


h.



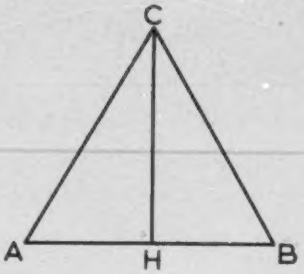
Consider: i. $\triangle RMW$ and $\triangle QMH$.

j. $\triangle WMX$ and $\triangle HMK$.


K
H
Q
c
b
a
M
a
b
c
R
W
X

5. A supplier wishes to telegraph a manufacturer for some parts in the form of triangular metal sheets. In addition to the thickness, kind of metal, and number of pieces wanted, what is the least he can say in order to specify the size and shape of the triangles? (Consider the possibility of more than one choice.)

6. Prove the following theorem: If the bisector of the angle opposite the base in an isosceles triangle intersects the base, it is perpendicular to the base.

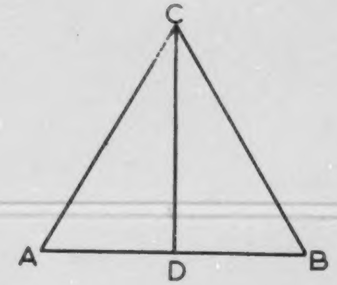


Restatement:

Given: $\triangle ABC$ with $AC = BC$ and H a point on $\overline{AB}$ such that $\angle ACH \cong \angle BCH$.

To prove: $\overline{CH} \perp \overline{AB}$.

7. Prove the theorem that the median from the vertex of an isosceles triangle is the bisector of the vertex angle.

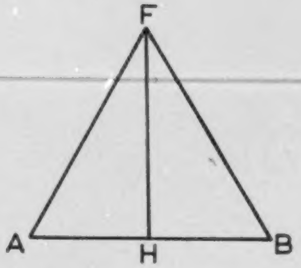


8. Prove the theorem: The bisector of the vertex angle of an isosceles triangle is the perpendicular bisector of the base.

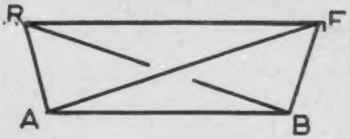
Restatement:

Given: $\triangle ABF$ with $AF = BF$ and H a point on $\overline{AB}$ such that $\overline{FH}$ bisects $\angle AFB$.

To prove: $\overline{AH} \cong \overline{BH}$ and $\overline{FH} \perp \overline{AB}$.

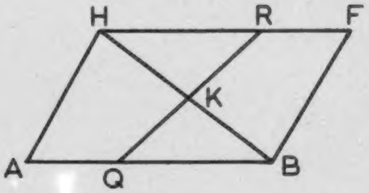


9. a. Given: In the figure,
$\overline{AF} \cong \overline{BR}$ and
$\overline{AR} \cong \overline{BF}$.
Prove: $\angle ARF \cong \angle BFR$.
(The gap in $\overline{RB}$ was left there so that the figure would not reveal whether or not $\overline{RB}$ intersects $\overline{AF}$.)

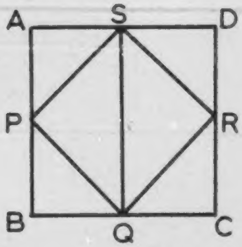


b. Do you need as part of the hypothesis that the figure lies in a plane?

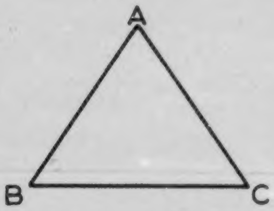
10. a. Given: In the figure,
$AH = FB$, $AB = FH$, and
$\overline{RQ}$ bisects $\overline{HB}$ in K.
Prove: $QK = RK$.
b. Is the figure necessarily planar?



11. Given: square ABCD with P, Q, R, S the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$, respectively.
Prove: $\triangle PQS \cong \triangle RQS$.



12. Point out why the following argument is circular, and thereby invalid.



Theorem: The base angles of an isosceles triangle are congruent.

Given: $\triangle ABC$ with $\overline{AB} \cong \overline{AC}$.

To prove: $\angle B \cong \angle C$.

Proof:

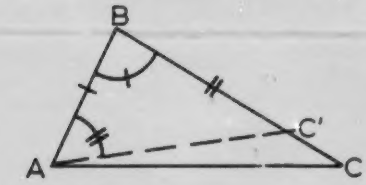
Statements	Reasons
1. $\overline{AB} \cong \overline{AC}$.	1. Given.
2. $\overline{AC} \cong \overline{AB}$.	2. Given.
3. $\overline{BC} \cong \overline{CB}$.	3. Identity.
4. $\triangle ABC \cong \triangle ACB$.	4. S.S.S.
5. $\angle B \cong \angle C$.	5. Definition of congruent triangles.

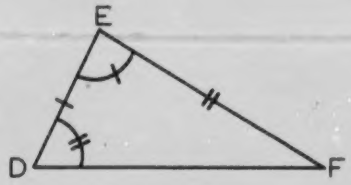
*13. Point out why the following argument is circular.

Theorem: Given a correspondence between two triangles (or between a triangle and itself). If two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

Given: $ABC \leftrightarrow DEF$, $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\angle ABC \cong \angle DEF$.

Prove: $\triangle ABC \cong \triangle DEF$.

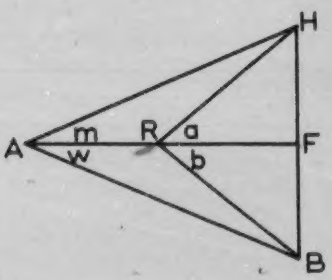




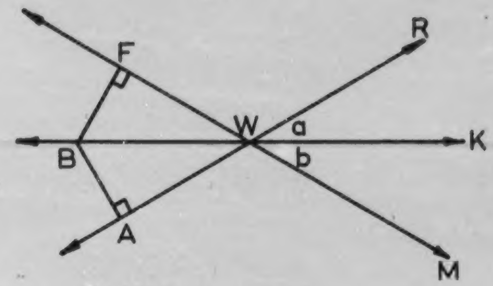
Proof:

Statement	Reasons
1. Let $\overrightarrow{AC'}$ be the ray on the same side of $\overleftrightarrow{AB}$ as $\overrightarrow{AC}$ such that $\angle BAC' \cong \angle EDF$, intersecting $\overleftrightarrow{BC}$ in C'.	1. Angle-Construction Postulate.
2. $\angle ABC' \cong \angle ABC$.	2. C' is on ray $\overrightarrow{BC}$, from step 1.
3. $\angle ABC \cong \angle DEF$.	3. Given.
4. $\angle ABC' \cong \angle DEF$.	4. Steps 2 and 3.
5. $\overline{AB} \cong \overline{DE}$.	5. Given.
6. $\angle BAC' \cong \angle EDF$.	6. Step 1.
7. $\triangle ABC' \cong \triangle DEF$.	7. A.S.A.
8. $\overline{BC'} \cong \overline{EF}$.	8. Corresponding parts.
9. $\overline{BC} \cong \overline{EF}$.	9. Given.
10. $\overline{BC'} \cong \overline{BC}$.	10. Steps 8 and 9.
11. $C' = C$.	11. Step 10 and the reason for Step 2.
12. $\triangle ABC \cong \triangle DEF$.	12. Steps 7 and 11.

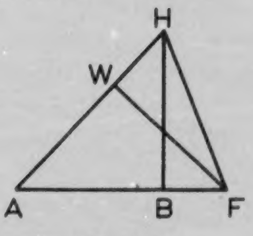
*14. If $\angle a \cong \angle b$ and $\angle m \cong \angle w$ in the figure, prove that $\overline{AF} \perp \overline{HB}$.



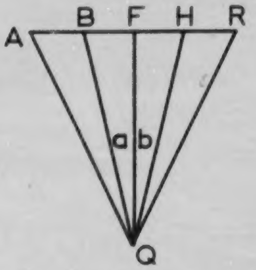
15. If in the figure $\overline{BF} \perp \overleftrightarrow{FM}$ at F, $\overline{BA} \perp \overleftrightarrow{AR}$ at A, and $m\angle a = m\angle b$, can you prove that FB = AB? If so, do so.



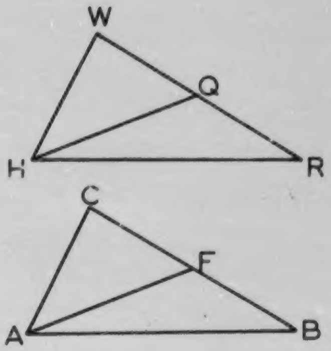
16. In $\triangle$ HAF, points B and W are on sides $\overline{AF}$ and $\overline{AH}$, respectively, and $\overline{FW} \perp \overline{AH}$, $\overline{HB} \perp \overline{AF}$, and AW = AB.
Prove: FW = HB.



17. If in the figure $\overline{FQ} \perp \overline{AR}$, $\overline{FQ}$ bisects $\angle AQR$, $\overline{BQ}$ bisects $\angle AQF$ and $\overline{HQ}$ bisects $\angle FQR$, prove that $\overline{BQ} \cong \overline{HQ}$.



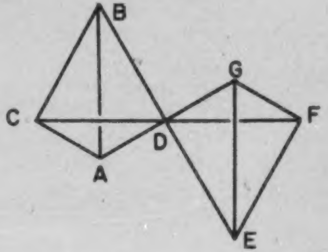
*18. In $\triangle ABC$ and $\triangle HRW$, AB = HR, AC = HW and median $\overline{AF} \cong$ median $\overline{HQ}$. On the basis of theorems you have had so far, can you show that $\triangle ABC \cong \triangle HRW$? If so, do so.



*19. Use the diagram for Problem 18 and suppose now it is given that AB = HR, BC = RW, and median $\overline{AF} \cong$ median $\overline{HQ}$. Can you prove that $\triangle ABC \cong \triangle HRW$? If so, do so.

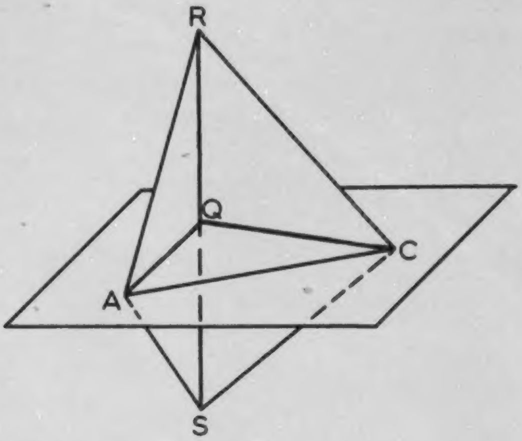
*20. Given: Points A, R, S, and C lie on Line L.
R lies between A and S.
S lies between R and C.
B and D do not lie on L.
AR = CS.
AB = CD.
BS = DR.
a. Prove that: $\angle BSA \cong \angle DRC$.
b. Need the points A, R, S, C, B, D be coplanar?

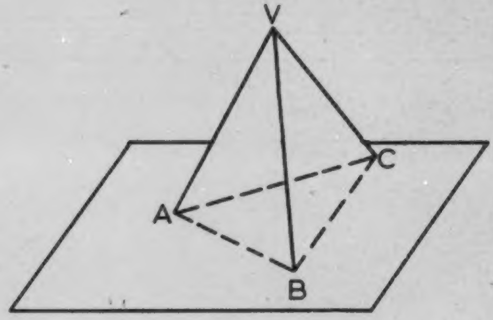
*21. In this figure D is the midpoint of $\overline{AG}$, $\overline{BE}$, and $\overline{CF}$.
Prove that $\triangle EFG \cong \triangle BCA$.



*22. Does the proof for Problem 21 hold even if the segments $\overline{BD}$, $\overline{AD}$, $\overline{CD}$ are not coplanar?

23. Given: In the figure, $\overline{AQ} \perp \overline{RS}$.
$\overline{RQ} \cong \overline{SQ}$.
$\overline{RC} \cong \overline{SC}$.
Prove that: $\angle RCA \cong \angle SCA$.





24. A tripod with three legs of equal lengths VA, VB, VC stands on a plane.

a. What can you say, if anything, about the distances AB, AC, BC? About the six angles ∠VAB, ∠VAC, ∠VBA, etc?

b. Answer part (a) if you are given also that the tripod legs make congruent angles with each other; that is, ∠AVB ≅ ∠BVC ≅ ∠AVC.

*25. a. Let $\overline{AR}$ and $\overline{BQ}$ bisect each other at M. Prove that AB = RQ and AQ = RB.

b. Now let $\overline{CX}$ also be bisected at M. How many pairs of congruent segments, as in (a) can you find?

c. You probably thought of $\overline{CX}$ as lying in the same plane as $\overline{AR}$ and $\overline{BQ}$. Is this necessary, or do your conclusions in (b) hold even if $\overline{CX}$ sticks out of the plane of $\overline{AR}$ and $\overline{BQ}$? Try to visualize the figure in the latter case, and either draw a picture or make a model.

26. Let △ABC be any triangle and D a point not in the plane of this triangle. The set consisting of the union of six segments $\overline{AB}$, $\overline{AC}$, $\overline{BC}$, $\overline{AD}$, $\overline{BD}$, $\overline{CD}$ we shall call a skeleton of a tetrahedron. Each of the six segments is called an edge of the tetrahedron, each of the four points A, B, C, D is a vertex, each triangle formed by three vertices is a face, each angle of a face is a face angle. Edges and faces of a tetrahedron were considered in Problem 11 of Problem Set 3-1c.

a. How many faces are there? How many face angles?

b. Two edges of a tetrahedron are opposite edges if they do not intersect. They are adjacent if they do intersect. If each pair of opposite edges are congruent, are any of the faces congruent? If each pair of adjacent edges are congruent, what kind of triangles are the faces?

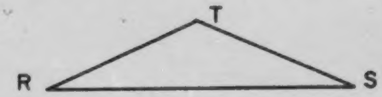
c. Construct an equilateral skeleton of a tetrahedron with toothpicks and quick-drying glue or with soda straws by threading string through them.

Review Problems for Chapter 5

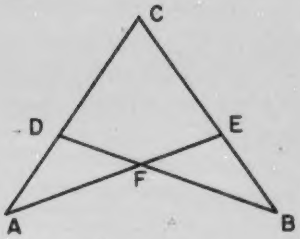
1. Complete:
If the vertices of two triangles correspond so that every pair of corresponding angles are ____________ and every pair of corresponding ____________ are congruent, then the correspondence is a ______________ between the two triangles.

2. Consider the set of abbreviations A.S.A., S.S.A., S.A.S., S.S.S., A.A.A.
   a. Which subsets are abbreviations of postulates in this chapter?
   b. Which subsets are abbreviations of theorems proved in this chapter?

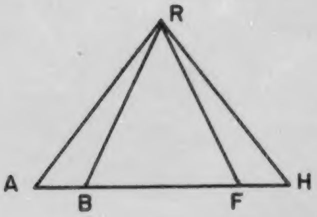
3. If $\triangle RST$ is isosceles with $RT = ST$, what correspondences are congruences between the triangle and itself?



4. Given $AF = BF$ and $DF = EF$, what would be the final reason in the most direct proof that $\triangle AFD \cong \triangle BFE$? That $\triangle AEC \cong \triangle BDC$?

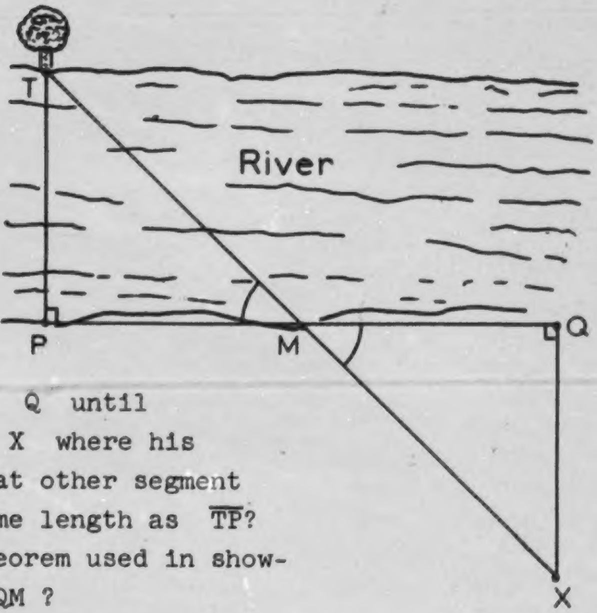


5. Given: In the figure $AR = RH$ and $AF = BH$.
Prove: $RB = RF$.

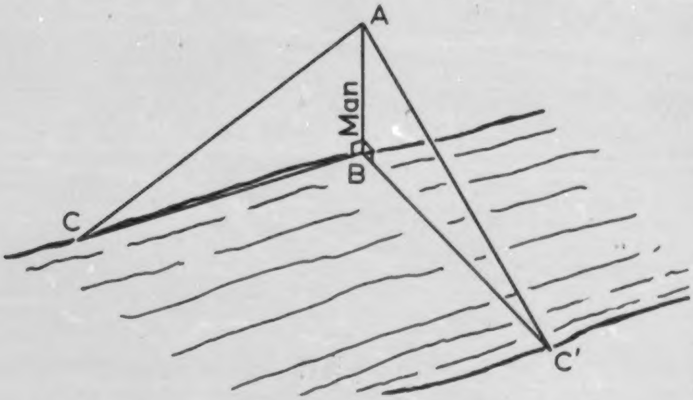


6. In the figure for Problem 5, if $RB = RF$ and $AB = HF$, prove that $AR = HR$.

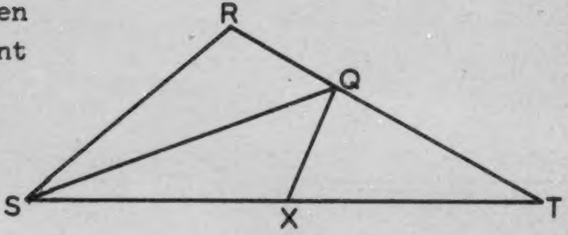
7. A person wishes to find the distance across a river. He does this by sighting a tree, T, on the other side opposite a point P, such that $\overline{PQ} \perp \overline{PT}$. Marking the midpoint, M, of $\overline{PQ}$, he paces a path perpendicular to $\overline{PQ}$ at Q until he determines the point X where his path meets line $\overleftrightarrow{TM}$. What other segment in the figure has the same length as $\overline{TP}$? What is the principal theorem used in showing that: $\triangle TPM \cong \triangle XQM$?



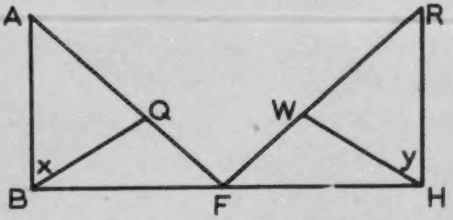
8. Napoleon's forces, marching into enemy territory, came upon a stream whose width they did not know. Although the engineers were in the rear, nevertheless, the impetuous commander demanded of his officers the width of the river. A young officer immediately stood erect on the bank and pulled the visor of his cap down over his eyes until his line of vision was on the opposite shore. He then turned and sighted along the shore and noted the point where his visor rested. He then paced off this distance along the shore. Was this distance the width of the river? What two triangles were congruent? Why?



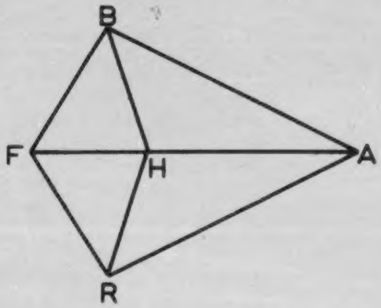
9. In $\triangle RST$: Point X lies between S and T, and SX = SR. Point Q lies between R and T, and $\overline{SQ}$ bisects $\angle S$. $\overline{QX}$ is drawn Find an angle congruent to $\angle R$, and establish the congruence.



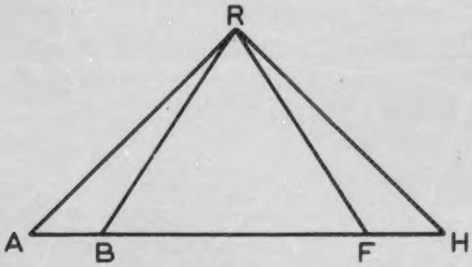
10. Given: The figure with $\overline{AB} \perp \overline{BH}$, $\overline{RH} \perp \overline{BH}$, $\angle x \cong \angle y$, QB = WH and F, the midpoint of $\overline{BH}$.
Prove: $\triangle BFQ \cong \triangle HFW$.



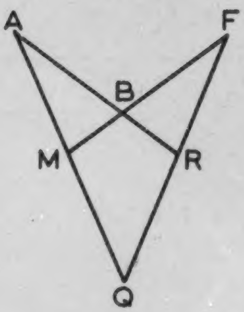
11. Given: In the figure, AB = AR and $\angle BAH \cong \angle RAH$.
Prove: FB = FR.



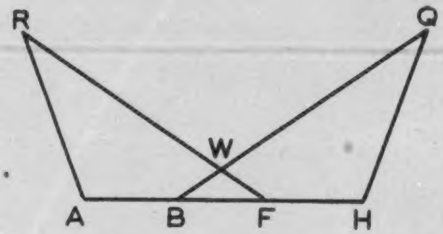
12. In this figure given that: AB = HF and RB = RF.
Prove: $\triangle AFR \cong \triangle HBR$.



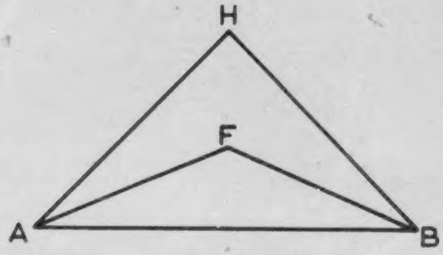
13. Given: In the figure,
AB = FB and MB = RB.
Prove: $\triangle AQR \cong \triangle FQM$.



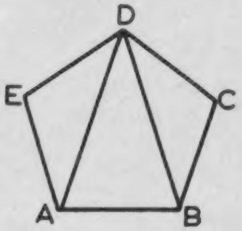
14. In this figure given
that B and F trisect* $\overline{AH}$,
$\angle A \cong \angle H$ and AR = HQ.
Prove: BW = FW.
*Trisect means to separate
into three congruent parts.



15. In this figure, given that
HA = HB,
$\overrightarrow{AF}$ bisects $\angle HAB$ and
$\overrightarrow{BF}$ bisects $\angle HBA$.
Prove: AF = BF.

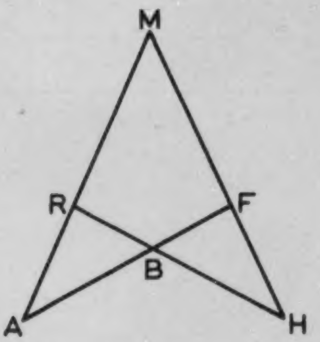


16. A polygon ABCDE has five sides of equal length and five angles of equal measure. Prove that $\angle DAB \cong \angle DBA$.



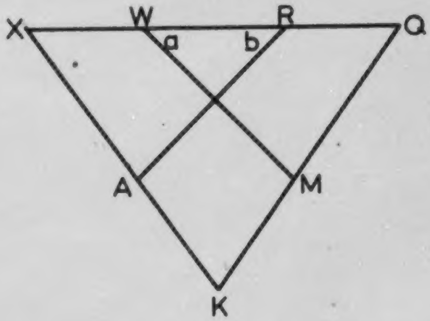
17. Prove: If two medians of a triangle are perpendicular to their respective sides, then the triangle is equilateral.

18. In this figure
$\overline{AB} \cong \overline{HB}$ and
$\overline{RB} \cong \overline{FB}$.
Prove: $\angle A \cong \angle H$ and
$\overline{AM} \cong \overline{HM}$.

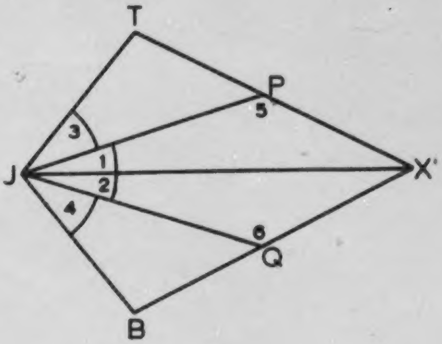


19. Prove that the bisectors of a pair of corresponding angles of two congruent triangles are congruent.

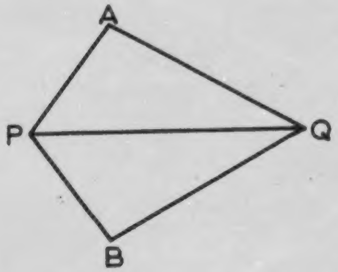
20. In this figure it is given that:
$XW = QR$,
$\angle a \cong \angle b$,
$\angle X \cong \angle Q$.
Prove: $KA = KM$.



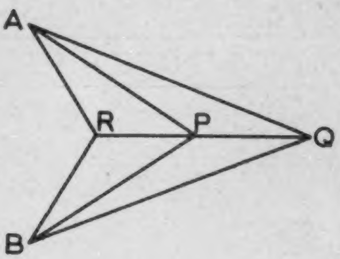
21. In this figure it is given that
$\angle 1 \cong \angle 2$, $\angle 3 \cong \angle 4$,
and $JT = JB$.
Prove: $\angle 5 \cong \angle 6$.



22. If $PA = PB$ and $QA = QB$ then $\angle APQ \cong \angle BPQ$. Will the same proof hold regardless of whether A is in the same plane as P, Q, and B ?

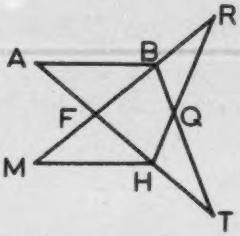


23. a. Prove: If PA = PB, QA = QB and R is on $\overleftrightarrow{PQ}$ as shown in the figure, then RA = RB.



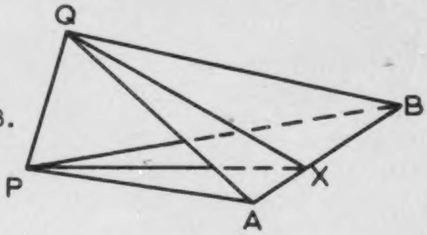
b. Must the five points be coplanar? Will the proof hold whether or not A is in the same plane as B, R, P, and Q?

*24. In this figure, points F and H trisect $\overline{AT}$, and points F and B trisect $\overline{MR}$. If AF = FB, is △ ABT ≅ △ MHR? Prove your answer.



25. If $\overleftrightarrow{RS}$ is perpendicular to each of three different rays, $\overrightarrow{RA}$, $\overrightarrow{RB}$, $\overrightarrow{RC}$ at R and RA = RB = RC, prove that SA = SB = SC. (Draw your own figure.)

*26. Let △ PAB and △ QAB lie in different planes but have the common side $\overline{AB}$. Let △ PAB ≅ △ QAB. Prove that if X is any point in AB then △ PQX is isosceles.

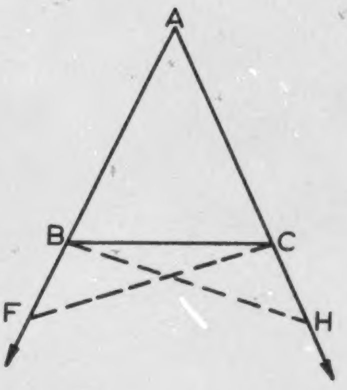


*27. Complete Euclid's proof of the theorem that the base angles of an isosceles triangle are congruent.

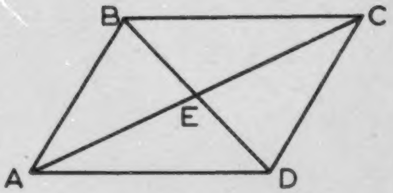
Given: AB = AC.

Prove: ∠ ACB ≅ ∠ ABC.

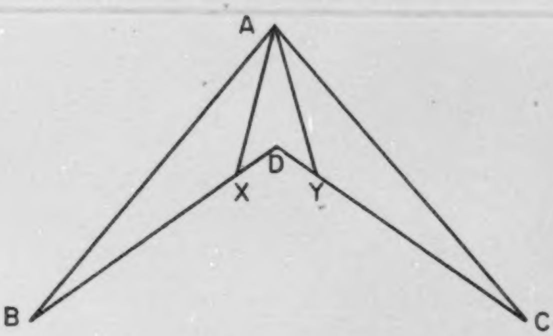
Construction: Take a point F with B between A and F, and a point H with C between A and H so that AH = AF. Draw $\overline{CF}$ and $\overline{BH}$.



*28. Given: The plane figure ABCD $AB = CD$, $AD = BC$.
Prove: $\overline{AC}$ and $\overline{BD}$ bisect each other.



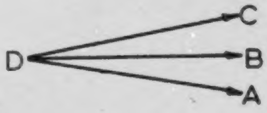
*29. Given: The figure ABCD with $AB = AC$, $DB = DC$, and $\angle BAX \cong \angle XAY \cong \angle CAY$.
Prove: $AX = AY$.



Chapters 1 to 5
REVIEW EXERCISES

Write the numbers from 1 to 80. Follow each with a "+" or a "-" to indicate whether you consider the statement true or false. True will mean "true under all conditions".

1. Every two rays intersect.
2. $\overline{AB}$ designates a line.
3. If $m\angle Q = 100$, then $\angle Q$ has no complement.
4. A line and a point not on it determine a plane.
5. If a point is in the interior of two angles of a triangle it is in the interior of the triangle.
6. If a line intersects a plane not containing it, then the intersection is one point.
7. The union of two half planes is a whole plane.
8. A point which belongs to the interior of an angle belongs to the angle.
9. If $\overline{AB} = \overline{CD}$, then either $A = C$ or $A = D$.
10. The intersection of every two half planes is the interior of an angle.
11. The interior of every triangle is convex.
12. It is possible to find two sets, neither of them convex, which have a union which is convex.
13. A ray has two end-points.
14. Experimentation is always the best way of reaching a valid conclusion.
15. Given four different points, no three of which are collinear, there are exactly six different lines determined by pairs of these points.
16. If $m\angle RST = m\angle XYZ$, then $\angle RST \cong \angle XYZ$.
17. In the figure the best way to name the angle formed by $\overrightarrow{DA}$ and $\overrightarrow{DC}$ is $\angle D$.



18. In this text "between" for points on a line is an undefined term.

19. The vertices of a triangle are non-collinear.

20. The intersection of two sets is the set of all elements that belong to one or both of them.

21. Every statement about geometric figures which is not a definition can be proved.

22. If $\triangle XYZ \cong \triangle CAB$, then $\angle A \cong \angle X$.

23. It is possible for two lines to intersect in such a way that three of the angles formed have measures 20, 70, and 20.

24. Each side of an angle is a ray.

25. All nouns which the text uses that relate to geometry are defined in the text.

26. The interior of an angle is a convex set.

27. If $m\angle ABC = 37$ and $m\angle DEF = 63$, then $\angle ABC$ and $\angle DEF$ are complementary.

28. If A is not between B and C, then C is between A and B.

29. $|m|$ is never a negative number.

30. If point Q is in the exterior of $\angle ABC$, then Q and C are on the same side of $\overleftrightarrow{AB}$.

31. The distance between two points is the absolute value of the sum of their coordinates.

32. The longest side of any triangle is called its hypotenuse.

33. If $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$ at point P (different from points A, B, C, D), then $m\angle APC + m\angle CPB + m\angle BPD + m\angle DPA = 360$.

34. Given a line, there is one and only one plane containing it.

35. A rational number is one which is the ratio of two integers.

36. Given two points on a line, a coordinate system can be chosen so that the coordinate of one point is zero and the coordinate of the other one is negative.

37. Two triangles are congruent if two sides and an angle of one are congruent to two sides and an angle of the other.

38. A collinear set of points is a line.

39. $x \leq 2x$.

40. The absolute value of every real number except zero is positive.

41. If $CD + CE = DE$, then D is between C and E.

42. If in $\triangle ABC$, $m \angle A = m \angle B = m \angle C$, then $AB = BC = AC$.

43. If, in a plane Z, $\overrightarrow{PT} \perp$ line L, $\overrightarrow{PQ} \perp$ line L, and P is on L, then $\overleftrightarrow{PT} = \overleftrightarrow{PQ}$.

44. From the statements (1) If q is false, then p is false, and (2) p is true, we can conclude that q is true.

45. The Ruler Postulate states that any unit can be reduced to inches.

46. If R is a point in the interior of $\angle XYZ$, then $m \angle XYR + m \angle ZYR = m \angle XYZ$.

47. There are certain points on a number scale which are not in correspondence with any number.

48. Every line is a collinear set of points.

49. $|-n| = n$.

50. The distance between two points is a positive number.

51. From the facts that $m \angle AOB = 20$ and $m \angle BOC = 30$ it can be concluded that $m \angle AOC = 50$.

52. A point on the edge of a half-plane belongs to that half-plane.

53. A line L in a plane E separates the plane into two convex sets.

54. The median of a triangle bisects the side to which it is drawn.

55. If two points lie in the same half-plane, then the line determined by them does not intersect the edge of that half-plane.

56. If two supplementary angles are congruent, each is a right angle.

57. The interior of an angle includes the angle itself.

58. Vertical angles have equal measures.

59. The sides of an angle are rays whose intersection is the vertex of the angle.

60. If $\angle C$ is supplementary to $\angle A$ and $m \angle A = 67$, then $m \angle C = 113$.

61. If two lines intersect, there are exactly two points of each which are contained by the other.

62. If two angles have equal measures the angles must be congruent.
63. From the statement (1) if p is true, then q is true, and (2) p is not true, we can conclude that q is false.
64. It has been proved in the first five chapters of this text that the sum of the measures of the angles of a triangle is 180.
65. The sides of a triangle are lines.
66. The midpoint of a segment separates it into two rays.
67. If two lines intersect so that the vertical angles formed are supplementary, then the measure of each angle is 90.
68. If $m \angle B = 93$, then $\angle B$ is acute.
69. For all numbers x, $|x| = x$.
70. The intersection of $\overrightarrow{AB}$ and $\overrightarrow{BA}$ is $\overline{AB}$.
71. In $\triangle ABC$ all points of $\overline{BC}$ are in the interior of $\angle A$.
72. If $\triangle ABC \cong \triangle BCA$, then $\triangle ABC$ is equilateral.
73. If $|x| = |y|$, then $x^2 = y^2$.
74. $\triangle ABC$ and $\triangle RFH$ which are in different planes are congruent if $AB = RF$, $BC = FH$ and $AC = RH$.
75. $\triangle ABC \cong \triangle MQT$ if $AB = QM$, $BC = TQ$ and $\angle Q \cong \angle B$.
76. Median $\overline{AB}$ in $\triangle ACE$ bisects $\angle A$.
77. If $x^2 = y^2$, then $|x| = |y|$.
78. If three points are on three different lines, the points are non-collinear.
79. There is no $\triangle ABC$ in which $\angle A = \angle B$.
80. Two points not on a plane are in opposite half-spaces determined by the plane if and only if the segment joining them intersects the plane.

Chapter 6

A CLOSER LOOK AT PROOF

6-1. How A Deductive System Works.

In Chapter 1 we tried to explain in general terms how our study of geometry was going to work. After the experience that you have had since then, you ought to be in a much better position to understand the explanation.

The idea of a set, the methods of algebra, and the process of logical reasoning, are things that we have been working with. The geometry itself is what we have been working on. We started with point, line and plane as undefined terms; and so far, we have used fifteen postulates. Sometimes, new terms have been defined by an appeal to postulates. (For example, the distance PQ was defined to be the positive number given by the Distance Postulate.) Sometimes definitions have been based only on the undefined terms. (For example a set of points is collinear if all points of the set lie on the same line.) But at every point we have built our definitions with terms that were, in some way, previously known. By now we have piled definitions on top of each other so often that the list is very long. And in fact, the length of the list is one of the main reasons why we had to be careful, at the outset, to keep the record straight.

In the same way, all the statements that we make about geometry are based ultimately on the postulates. Sometimes we have proved theorems directly from postulates, and sometimes we have based our proofs on theorems that were already proved. But in every case, the chain of reasoning can be traced backward to the postulates.

You might find it a good idea, at this point, to reread the second half of Chapter 1. It will seem much clearer to you now than it did the first time. It is much easier to look back, and understand what you have done, than to understand an explanation of what you are about to do.

6-2. Indirect Proof.

We remarked in Chapter 1 that the best way to learn about logical reasoning is by doing some of it. There is one kind of proof, however, that may require some additional discussion. For Theorem 3-1, we used what is called an indirect proof. The theorem and its proof were as follows:

"Theorem 3-1. Two different lines intersect in at most one point.

Proof: It is impossible for two different lines to intersect in two different points P and Q. This is impossible because by Postulate 1 there is only one line that contains both P and Q."

Probably this was the first time that you had seen this kind of reasoning used in mathematics, but you must have encountered the same sort of thing, many times, in ordinary conversation. Both of the following remarks are examples of indirect proofs:

(1) "It must be raining outside. If it were not raining, then those people coming in the door would be dry, but they are soaking wet."

(2) "Today must not be the right day for the football game. If the game were today, then the stadium would be full of people, but you and I are the only ones here."

In each of these cases, the speaker wants to show that his first statement is true. He starts his proof by supposing that the thing he wants to prove is wrong; and then he observes that this leads to a conclusion which contradicts a known fact. In the first case, the supposition is that it is not raining; this leads to the conclusion that the people coming in would be dry, which contradicts the known fact that these people are wet; and therefore it is raining, after all. Similarly, in the second case the assumption that the game is today leads to a contradiction of the known fact of the empty stadium.

In the proof of Theorem 3-1, the supposition is that some two different lines intersect in two points. By Postulate 1, this leads to the conclusion that the lines aren't different after all. Therefore the supposition is wrong, and this means that the theorem is right.

Problem Set 6-2a

1. For the sake of argument accept each of the following assumptions and then give a logical completion for each conclusion.
   a. Assumption: Only men are color blind.
      Conclusion: My mother --------.
   b. Assumption: All men are left-handed.
      Conclusion: My brother --------.
   c. Assumption: The only thing that makes Jane ill is hot chocolate. Jane is ill.
      Conclusion: Jane --------.
2. Which of the following arguments are indirect?
   a. The temperature outside must be above 32° F. If the temperature were not above 32°, then the snow would not be melting. But it is melting. Therefore, the temperature must be above 32°.
   b. That movie must be very entertaining. If it were not very entertaining, then only a few people would go to see it. But large crowds are going to see it. Therefore, it must be very entertaining.
   c. The air-conditioning in this building must not be working correctly. If it were working correctly, then the temperature would not be so high. But the temperature is uncomfortably high. Therefore, the air-conditioning is not working correctly.
3. Mrs. Adams purchased a set of knives, forks, and spoons advertised as a stainless steel product. After using the set for several months, she found that the set was beginning to rust. She thereupon decided that the set was not stainless steel and returned it for refund.

   In this example of indirect proof identify (1) the statement to be proved, (2) the supposition made, (3) the conclusion resulting from the supposition, and (4) the known fact contradictory to (3).

4. What conclusions can you draw from the following hypothesis in which x, y and z stand for different statements?
   If x is true, then y is true.
   If y is true, then z is true.
   x is true.
5. Suppose you have the following data:
   If w is true, then v is true.
   If u is true, then w is true.
   If x is true, then u is true.
   v is not true.
   What conclusions can you draw? Did you use indirect reasoning at any point?
6. What conclusion follows from the following data?
   (1) Nobody is allowed to join the swimming club unless he can play the piccolo.
   (2) No turtle can play the piccolo.
   (3) Nobody is allowed to wear striped trunks in the club pool unless he is a member of the swimming club.
   (4) I always wear striped trunks in the club pool.
   (Hint: This problem becomes easier if you convert it to if-then form, as in several preceding problems. For example, let A be "someone is a member of the swimming club", let B be "someone can play the piccolo", etc.)
7. If A is green, then B is red.
   If A is blue, then B is black.
   If B is red, then Y is white.
   a. A is green, so B is ______ and Y is ________.
   b. B is black. Is it possible to draw a conclusion concerning A? If so, what conclusion?
8. Prove that the bisector of any angle of a scalene triangle cannot be perpendicular to the opposite side.

Let us now prove the other theorems of Chapter 3. For convenience, we first restate the postulates on which these proofs are based:

Postulate 1. Given any two different points, there is exactly one line which contains both of them.

Postulate 5. a. Every plane contains at least three non-collinear points. b. Space contains at least four non-coplanar points.

Postulate 6. If two points lie in a plane, then the line containing these points lies in the same plane.

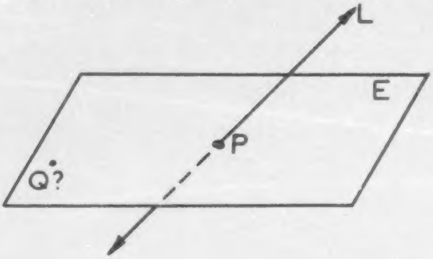
Postulate 7. Any three points lie in at least one plane, and any three non-collinear points lie in exactly one plane. More briefly, any three points are coplanar, and any three non-collinear points determine a plane.

Theorem 3-2. If a line intersects a plane not containing it, then the intersection is a single point.

Proof: By hypothesis, we have a line L and a plane E, and

(1) L intersects E in at least one point P, and

(2) E does not contain L.



We are going to give an indirect proof of the theorem and therefore we start by supposing that the conclusion is false. Thus our supposition is that

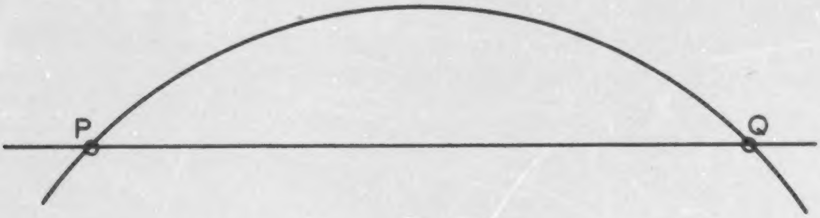
(3) L intersects E in some other point Q.

To give an indirect proof, we need to show that our supposition contradicts a known fact. And it does: If P and Q lie in E, it follows by Postulate 6 that the line containing them lies in E. Therefore

(4) L lies in E.

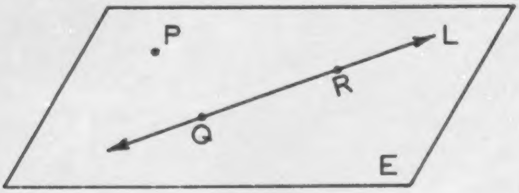
This contradicts (2). Therefore the supposition (3) is impossible. Therefore Theorem 3-2 is true.

Notice that the figures that we use to illustrate indirect proofs look peculiar. In the figure for Theorem 3-2, we have indicated a point Q, merely to remind ourselves of the notation of the proof. The proof itself shows that no such point Q can possibly exist. In fact, the figures for indirect proofs always look ridiculous, for a good reason: they are pictures of impossible situations. If we had drawn a figure for Theorem 3-1, it would have looked even worse, perhaps like this:



This is a picture of an impossible situation in which two different lines intersect in two different points.

Theorem 3-3. Given a line and a point not on the line, there is exactly one plane containing both of them.



Proof: By hypothesis we have a line L and a point P not on L. By the Ruler Postulate we know that every line contains infinitely many points, and so L contains two points Q and R. By Postulate 7 there exists a plane E which contains P, Q, and R. Since by Postulate 6, E contains L, we have shown that there exists a plane E containing both L and P.

At this point we actually have proved only half of the theorem, since Theorem 3-3 says there is exactly one such plane. It remains to prove that no other plane containing L and P exists. We do this by indirect proof

Suppose that there is another plane E' containing L and P. Since by Postulate 1 L is the only line containing Q and R, we know that Q and R, as well as P, lie in E'. This contradicts Postulate 7 which says that exactly one plane contains three non-collinear points. Since E was established as a plane containing P, Q and R, E' can not exist, and E is the only plane containing L and P.

The two parts of the proof of Theorem 3-3 bring up the distinction between existence and uniqueness. The first half of the proof shows the existence of a plane E containing L and P. This leaves open the possibility that there may be more than one such plane. The second half of the proof shows the uniqueness of the plane. When we prove existence, we show that there is at least one object of a certain kind. When we prove uniqueness we show that there is at most one. If we prove both existence and uniqueness, this means that there is exactly one.

For example, for the fleas on a stray dog, we can usually prove existence, but not uniqueness. (It is a very lucky dog that has only one flea.) For the eldest daughters of a given woman, we can obviously prove uniqueness, but not necessarily existence; some women have no daughters at all. For the points common to two different segments, we don't necessarily have either existence or uniqueness; the intersection may contain many points, or exactly one point, or no points at all.

The phrase "one and only one" is often used instead of "exactly one" since it emphasizes the double nature of the statement.

The following theorem breaks up into two parts in exactly the same way:

Theorem 3-4. Given two intersecting lines, there is exactly one plane containing them.

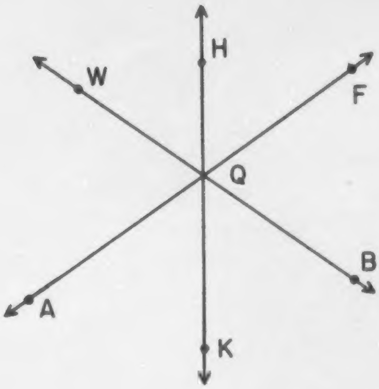
For variety we give the proof in double-column form. Note the two parts and the way we handle the indirect proof in the second part.

Proof: We have given the lines L_1 and L_2, intersecting in the point P.

Statements	Reasons
1. L_1 contains a point Q, different from P.	1. By the Ruler Postulate, every line contains infinitely many points.
2. Q is not on L_2.	2. Theorem 3-1.
3. There is exactly one plane E, containing L_2 and Q.	3. Theorem 3-3.
4. E contains L_1.	4. By Postulate 6, since E contains P and Q.
5. Suppose that another plane F also contains L_1 and L_2.	
6. F contains Q.	6. Q is on L_1.
7. E and F each contain L_2 and Q.	7. Steps 3 and 4, and 5 and 6.
8. E is the only plane containing L_1 and L_2.	8. Step 7 contradicts Theorem 3-3.

Problem Set 6-2b

1. Is a triangle necessarily a plane figure? Explain.
2.

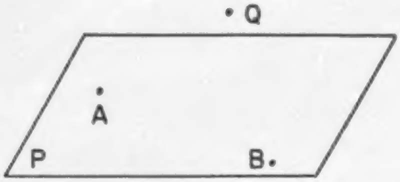


Theorem 3-4 says, in effect, "Two intersecting lines determine a plane". How many different planes are determined by pairs of intersecting lines in this figure? Assume that the three lines are not all in the same plane. List each plane by naming the two intersecting lines that determine it.

3. How many different planes are determined by pairs of the four different lines $\overleftrightarrow{AQ}$, $\overleftrightarrow{BQ}$, $\overleftrightarrow{CQ}$, and $\overleftrightarrow{DQ}$, no three of which lie in the same plane? List the planes by naming for each the two intersecting lines that determine it.

4. If, in a plane Z, $\overrightarrow{PT} \perp$ line L and $\overrightarrow{PQ} \perp$ line L, what conclusion can you draw regarding $\overleftrightarrow{PQ}$ and $\overleftrightarrow{PT}$?

5.



As indicated in this figure, A and B lie in plane P. Q lies above plane P. Does line $\overleftrightarrow{AB}$ lie entirely in P? Quote a postulate or theorem to support your conclusion. There is a second plane implicit in the situation. Name it by the three points which determine it. What is the intersection of these two planes? At what point will $\overleftrightarrow{QB}$ intersect plane P?

6. If A, B, C, D are four non-collinear points, list all the planes determined by subsets of A, B, C, D.

6-3. Theorems about Perpendiculars.

Some of the basic theorems about perpendicular lines are good examples of existence, uniqueness, and indirect proofs.

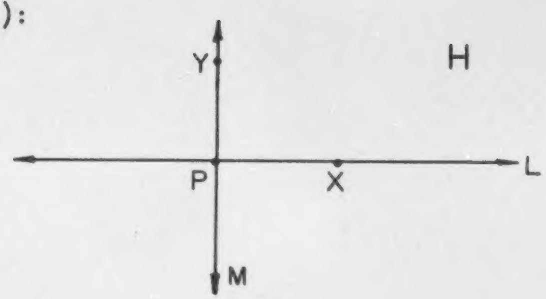
Theorem 6-1. In a given plane, through a given point of a given line of the plane, there passes one and only one line perpendicular to the given line.

Given: E is a plane, L a line in E, and P a point of L.

To prove: (1) There is a line M in E, such that M contains P and $M \perp L$;

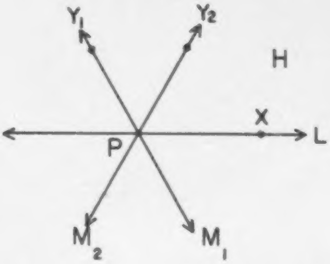
(2) There is at most one line in E, containing P and perpendicular to L.

Proof of (1):



Let H be one of the two half-planes in E that have L as an edge, and let X be a point of L, different from P. By the Angle Construction Postulate, there is a point Y of H, such that $\angle XPY$ is a right angle. Let M be the line $\overleftrightarrow{PY}$. Then $M \perp L$. Thus we have proved that there is at least one line satisfying the conditions of the theorem.

Proof of (2): We now need to prove that there is at most one such line. Suppose that there are two of them, M_1 and M_2. Let X be a point of L, different from P.



Then the lines M_1 and M_2 contain rays $\overrightarrow{PY_1}$ and $\overrightarrow{PY_2}$ lying in the same half-plane H having L as its edge. By definition of perpendicular lines, one of the angles determined by L and M_1 is a right angle, and by Theorem 4-8 all four of these angles are right angles. Thus $m\angle XPY_1 = 90$. Similarly, $m\angle XPY_2 = 90$. But this contradicts the Angle Construction Postulate, which says that there is only one ray $\overrightarrow{PY}$, with Y in H, such that $m\angle XPY = 90$. This contradiction means that our assumption of two perpendiculars M_1 and M_2 must be false, which proves the second half of the theorem.

The condition "in a given plane" is an important part of the statement of this theorem. If this condition were omitted the first (existence) part of the theorem would still be true but the second (uniqueness) part would not. This is easily seen by thinking of the relation between the spokes of a wheel and the axle. Thus leaving out this condition would give us an example of a geometric existence theorem with no corresponding uniqueness theorem. The opposite situation, a uniqueness theorem with no corresponding existence theorem, has already been considered in this chapter. Can you identify it?

Definition. The perpendicular bisector of a segment, in a plane, is the line in the plane which is perpendicular to the segment and contains the mid-point.

Every segment has exactly one mid-point, and through the mid-point there is exactly one perpendicular line in a given plane. Thus, for perpendicular bisectors in a given plane, we have both existence and uniqueness.

The following theorem gives a useful characterization of the points of a perpendicular bisector:

Theorem 6-2. The perpendicular bisector of a segment, in a plane, is the set of all points of the plane that are equidistant from the end-points of the segment.

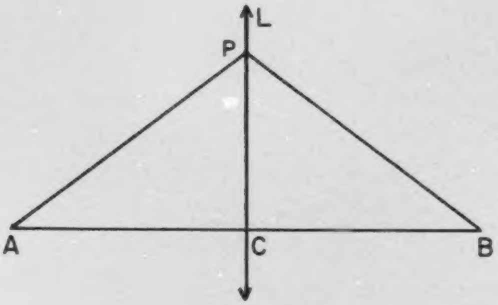
Restatement: Let L be the perpendicular bisector of the segment $\overline{AB}$ in a plane E and let C be the mid-point of $\overline{AB}$. Then

(1) If P is on L, then $PA = PB$, and

(2) If P is in E, and $PA = PB$, then P is on L.

Notice that the restatement makes it plain that the proof of the theorem will consist of two parts. In the first part we prove that every point of the perpendicular bisector satisfies the characterization, that is, is equidistant from the end-points of the segment. But the theorem says that the perpendicular bisector is the set of all such points. To prove this, then, we must also show that every such point, characterized by being equidistant from the end-points of the segment, is on the perpendicular bisector. This last is the second part of the restatement.

Proof of (1): Given a point P of L. If P lies on $\overleftrightarrow{AB}$, then P = C, and this means that PA = PB by the definition of mid-point of a segment. If P is not on the line $\overleftrightarrow{AB}$, then

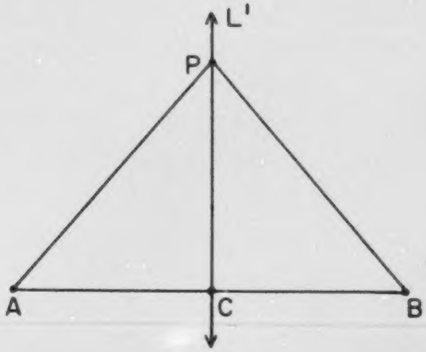


PC = PC by identity, and, by hypothesis, CA = CB and $\angle PCA \cong \angle PCB$. Hence by the S.A.S. Postulate,

$$\triangle PCA \cong \triangle PCB.$$

Therefore PA = PB, which was to be proved.

Proof of (2): Given that P lies in the plane E and PA = PB. If P is on $\overleftrightarrow{AB}$, then P is the mid-point C of $\overline{AB}$, and so P is on L. If P is not on $\overleftrightarrow{AB}$, let L' be the line $\overleftrightarrow{PC}$:



then PC = PC, CA = CB, and PA = PB. (Why?) By the S.S.S. Theorem,

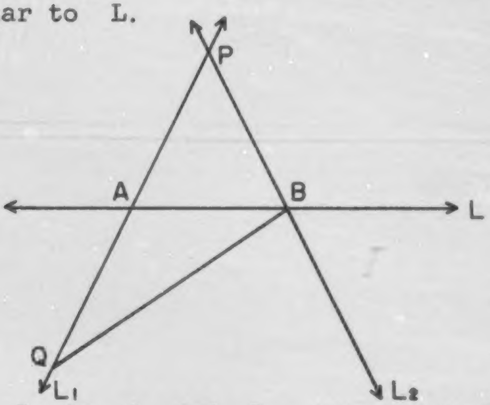
$$\triangle PCA \cong \triangle PCB.$$

Therefore $\angle PCA \cong \angle PCB$. Therefore, by definition, $L' \perp \overline{AB}$ and so L' is the perpendicular bisector of $\overline{AB}$. Therefore, by Theorem 6-1, $L' = L$, and P is on L, which was to be proved.

Next we prove the analog of Theorem 6-1 for the case in which the given point is not on the given line. Since the proof is considerably more complicated than that of Theorem 6-1, we will state and prove the existence and the uniqueness parts as separate theorems. Because it is the simpler, we start with uniqueness.

Theorem 6-3. Through a given external point there is at most one line perpendicular to a given line.

Proof: Like most uniqueness proofs, this is an indirect one. Suppose L_1 and L_2 are distinct lines through point P, each perpendicular to L.



Let L_1 intersect L in A and L_2 intersect L in B. Since the lines are distinct and both go through P we must have $A \neq B$ (Theorem 3-1).

On the ray opposite to $\overrightarrow{AP}$ take AQ = AP (Point Plotting Theorem). Then AQ = AP, AB = AB, $m\angle PAB = m\angle QAB = 90$, and so $\triangle QAB \cong \triangle PAB$ by the S.A.S. Postulate.

It follows that

$$m\angle QBA = m\angle PBA = 90 ,$$

and so $\overleftrightarrow{BQ} \perp L$. This contradicts Theorem 6-1, which says that there is only one perpendicular to L at B lying in the plane containing L and L_1. Hence our supposition that there could be two perpendiculars to L through P is false.

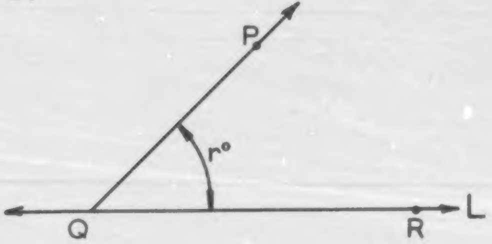
Corollary 6-3-1. At most one angle of a triangle can be a right angle.

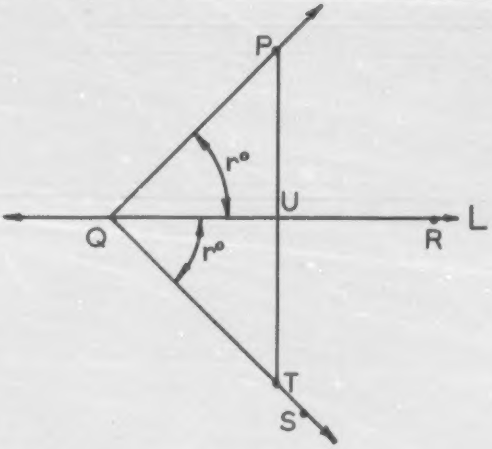
For if in $\triangle ABC$, $\angle A$ and $\angle B$ were both right angles we would have two perpendiculars from C to $\overleftrightarrow{AB}$.

Definitions. A right triangle is a triangle one of whose angles is a right angle. The side opposite the right angle is the hypotenuse; the sides adjacent to the right angle are the legs.

Theorem 6-4. Through a given external point there is at least one line perpendicular to a given line.

Restatement: Let L be a line, and let P be a point not on L. Then there is a line perpendicular to L and containing P.





First we will explain how the perpendicular can actually be constructed, on paper, using a ruler and a protractor. From the method of construction, it will be clear how the theorem can be proved from the postulates.

Step 1. Let Q and R be any two points of the line L. Measure the angle $\angle PQR$.

Step 2. Using the protractor, construct an angle $\angle RQS$, with the same measure as $\angle PQR$, taking S on the opposite side of the line L from P.

Step 3. Measure the distance QP. Take a point T on $\overrightarrow{QS}$, such that $QT = QP$.

Step 4. Now draw the line $\overleftrightarrow{TP}$. This is the perpendicular that we were looking for. For the reasons, see the proof below. First, however, you should try this construction with your ruler and protractor, and try to see for yourself why it works.

Let us now write down the proof in the double-column form. Each of the first few statements on the left corresponds to one of the things that we were doing with our drawing instruments.

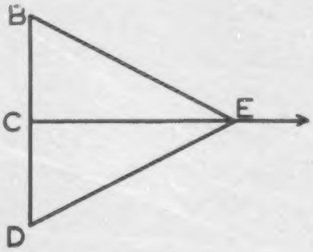
Statements	Reasons
1. L contains two points Q and R.	1. The Ruler Postulate.
2. There is an angle $\angle RQS$, congruent to $\angle RQP$, with S and P on different sides of L.	2. The Angle Construction Postulate.
3. There is a point T of the ray $\overrightarrow{QS}$, such that $QT = QP$.	3. The Point Plotting Theorem.
4. T and P are on opposite sides of L.	4. P and S are on opposite sides of L, and S and T are on the same side of L.
5. $\overline{TP}$ intersects L, in a point U.	5. Definition of opposite sides.
6. $\Delta PQU \cong \Delta TQU$.	6. Statement 2, statement 3, and the S.A.S. Postulate.
7. $\angle QUP \cong \angle QUT$.	7. Definition of a congruence between triangles.
8. $\angle QUP$ is a right angle.	8. Definition of right angle.
9. $\overleftrightarrow{PT} \perp L$.	9. Definition of perpendicularity.

This proof somewhat resembles the proof of the S.S.S. Theorem (Theorem 5-6). Like this earlier theorem it has several cases, only one of which (that in which U and R lie on the same side of Q) is completely covered by the above proof.

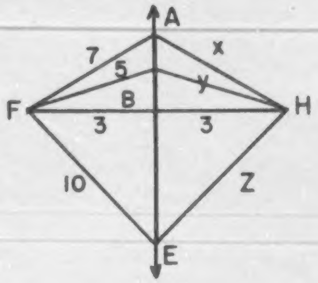
The modifications necessary for the other two cases (U = Q and Q is between R and U) are left as exercises for the student.

Problem Set 6-3

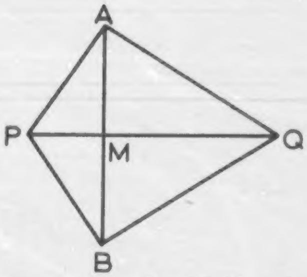
1. If $BC = DC$ and $\overleftrightarrow{EC} \perp \overleftrightarrow{BD}$, prove without the use of congruent triangles that $EB = ED$.



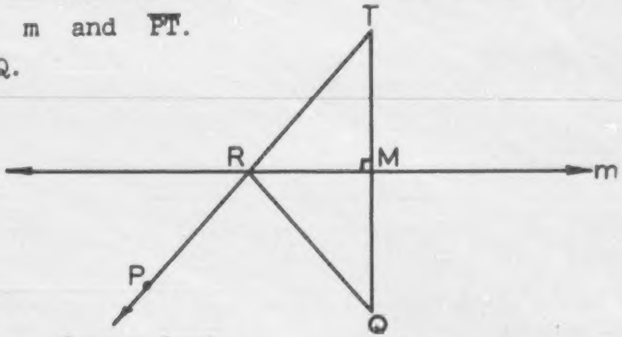
2. If $\overleftrightarrow{AE} \perp \overline{FH}$ at B as shown in the figure, with lengths of segments as indicated, find x, y and z.



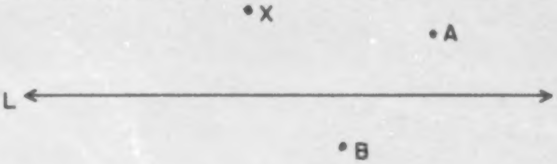
3. Given: $PA = PB$, M is the midpoint of $\overline{AB}$, and Q is on line $\overleftrightarrow{PM}$ as shown in the figure.
Prove: $QA = QB$.
(Use paragraph proof.)



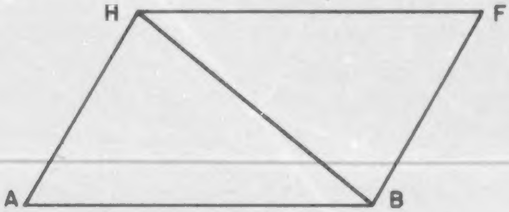
4. Given: The line m is the perpendicular bisector of the segment $\overline{QT}$. P is on the same side of m as Q. R is the intersection of m and $\overline{PT}$.
Prove: $PT = PR + RQ$.



5. Copy the figure below. Following the steps outlined in the text construct perpendiculars from A, B and X to line L.



6. Copy the figure. Using ruler and protractor construct perpendiculars from A and F to $\overline{HB}$.



7. Does Theorem 6-4 state the existence of a unique perpendicular to a line from a point off the line? If we confine our thinking to a plane, does Theorem 6-1 state the existence of a unique perpendicular to a line through a point on the line?

*8. Given isosceles triangle ABC with AC = BC and bisectors $\overrightarrow{AD}$ and $\overrightarrow{BE}$ of $\angle A$ and $\angle B$. $\overline{AD}$ and $\overline{BE}$ intersect at point F. Prove that $\overleftrightarrow{CF}$ is perpendicular to $\overline{AB}$. (It is not necessary to use any congruent triangles in your proof.)

*9. One diagonal of a quadrilateral bisects two angles of the quadrilateral. Prove that it bisects the other diagonal.

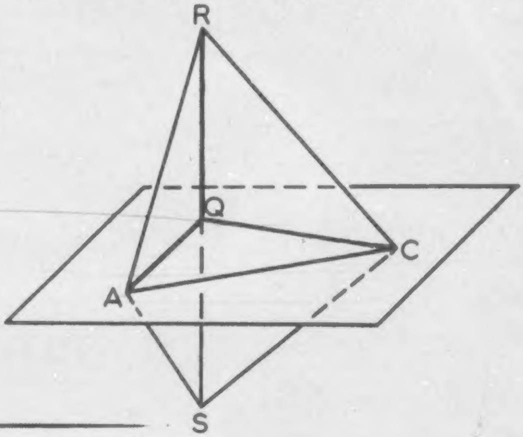
*10. In this figure given:

$RC = SC$,

Q is midpoint of $\overline{RS}$,

$\angle RCA \cong \angle SCA$.

Prove: $\overline{AQ} \perp \overline{RS}$.



6-4. Introducing Auxiliary Sets into Proofs.

You probably noticed that in proving some theorems, most recently, Theorems 6-2 and 6-4, we introduced certain points, rays and segments into the figure in addition to those specified in the theorem. Possibly two questions concerned you:

1. How can we justify introducing such additional sets into proofs on the basis of our postulates?

2. How do we know which of these sets, if any, should be introduced into the proof of a theorem?

The first question is easy to answer. In working with theorems we usually are concerned with various relationships among certain points, lines, planes and subsets of these, and as a practical matter in proving theorems, we choose certain planes or lines and certain points on them. Frequently we do not concern ourselves with justifying this procedure. For example, if we are given a line we may immediately name it $\overleftrightarrow{PQ}$. When asked to give a reason, however, we can refer to the Ruler Postulate, which says that a line contains infinitely many points, and thereby the two points P and Q exist. Similarly, given two points A and B we may talk about $\overleftrightarrow{AB}$ with complete confidence since it stands for a line whose existence and uniqueness are guaranteed by Postulate 1. (See Section 6-2.)

The careful concern over justifying existence and uniqueness becomes especially important when we introduce into the proof certain points, lines, segments, and so on, not accounted for by the theorem being proved. Certainly we can not have these sets in our proofs if they do not exist under the conditions of our geometry, except, of course, in an indirect proof, where the object is to show they can't exist.

In the table below we list the postulates and theorems occurring so far which may be used, appropriately, to introduce auxiliary sets into proofs.

Geometric Set	Existence	Uniqueness
1. Point.	Postulates 3 and 5.	Theorems 2-4, 3-1, 3-2.
a. Midpoint.	Theorem 2-5.	Theorem 2-5.
2. Line.	Postulates 1 and 8.	Postulates 1 and 8.
a. Perpendicular at point on line, in a plane.	Theorem 6-1.	Theorem 6-1.
b. Perpendicular bisector, in a plane.	Theorems 2-5 and 6-1	Theorems 2-5 and 6-1.
c. Perpendicular from point not on line.	Theorem 6-4.	Theorem 6-3.
3. Plane.	Postulate 7. Theorems 3-3 and 3-4.	Postulate 7. Theorems 3-3 and 3-4.
4. Ray as used in angle measure.	Postulate 12.	Postulate 12.
a. Bisector of an angle.	Theorem 5-3.	Theorem 5-3.
5. Segment.	Postulate 1 and Definition of segment.	Postulate 1 and Definition of segment.

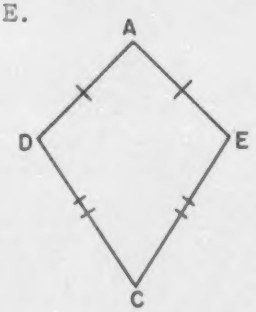
From this table you may see that you already know a lot about the nature of our three basic undefined terms.

The answer to the second question presents a problem quite different from the answer to the first. Getting to know when to introduce auxiliary sets into a proof is largely part of the process of learning to reason logically. It requires considerable practice. Let's try an example to see how this works.

Example 1.

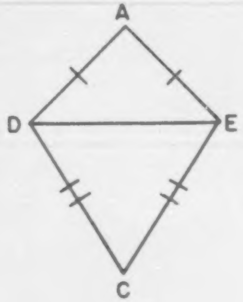
Given: The plane figure with AD = AE and CD = CE.

To prove: $\angle D \cong \angle E$.



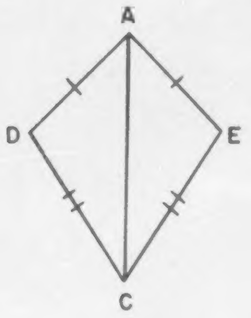
Since all of our postulates and theorems concerning congruence have dealt with triangles, it seems reasonable that our figure should show some triangles. We can accomplish this easily by introducing either $\overline{AC}$ or $\overline{DE}$.

Suppose we introduce $\overline{DE}$ so that our figure looks like this:



This allows us to complete the proof, since $m\angle ADE = m\angle AED$ and $m\angle CDE = m\angle CED$ gives us $m\angle ADC = m\angle AEC$ by the Angle Addition Postulate.

Had we introduced $\overline{AC}$ instead of $\overline{DE}$, our proof, in two-column form this time, would have looked like this:



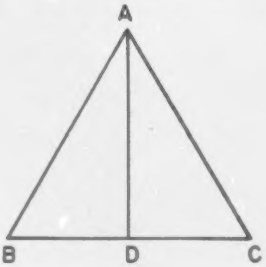
Proof:

Statements	Reasons
1. Introduce $\overline{AC}$.	1. Postulate 1 and Definition of segment.
2. $AC = AC$.	2. Identity.
3. $AD = AE$ and $CD = CE$.	3. Given.
4. $\triangle ADC \cong \triangle AEC$.	4. S.S.S. Theorem.
5. $\angle D \cong \angle E$.	5. Definition of congruent triangles.

Each of the solutions to Example 1 is correct. The choice of which one you use is up to you. But it is worth noting that in many problems where a choice exists, the choice you make will determine the degree of difficulty of the proof. It is helpful to think through each solution before writing one down formally.

An important aspect of learning what to introduce in a proof can be illustrated if we remove from the hypothesis of Example 1 the condition that the figure is a plane figure. If D is not coplanar with A, E, and C, at least one of the solutions does not hold. Does either solution hold? If one does, which one?

One final word of warning before you begin to introduce auxiliary sets into your proofs. In answering Question 1 we were careful to say that each such step must be justifiable, that is, that every point, line, plane, and so on must exist under our postulates. Students often make the mistake of not recognizing this. For example, you might think you could prove the statement "All angles are congruent" by the following argument.



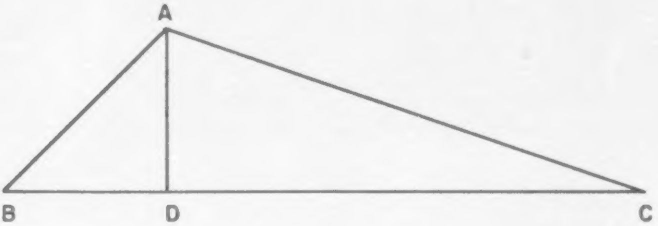
Example 2.

Given any $\triangle ABC$, prove that $\angle B \cong \angle C$.

Proof: In $\triangle ABC$ introduce $\overline{AD}$, bisecting $\angle A$ and perpendicular to $\overline{BC}$.

Then $\angle BAD \cong \angle CAD$ by definition of the bisector of an angle, $AD = AD$ by identity, and $\angle BDA \cong \angle CDA$ by the definition of perpendicular and the fact that all right angles are congruent. Therefore $\triangle BAD \cong \triangle CAD$ by A.S.A., making $\angle B \cong \angle C$.

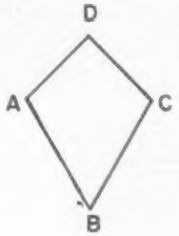
It does not take long to see the serious error of this so-called proof. The segment $\overline{AD}$, as angle bisector and the perpendicular to the base, does not exist under our postulates. Moreover, the figure makes $\triangle ABC$ appear to be isosceles and thus makes $\overline{AD}$ appear as introduced above. Were the figure like this,



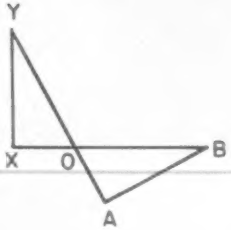
you certainly would not consider using $\overline{AD}$ as it is used. This leads us once more to say that the figure is merely a convenience to aid you in thinking through your reasoning in logical and carefully chosen words.

Problem Set 6-4

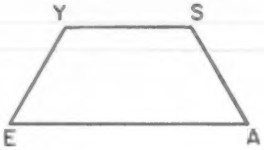
1. Given: A, B, C and D are coplanar. AD = CD. m∠A = m∠C. Prove: AB = CB. Does the proof work if A, B, C, D are not coplanar?



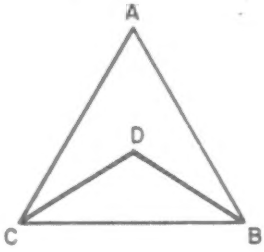
2. Given: XY = AB, AY = XB. Prove: △XOY ≅ △AOB.



3. Given: E, A, S and Y are coplanar. ∠E ≅ ∠A, $\overline{YE} \cong \overline{SA}$. Prove: ∠Y ≅ ∠S.



4. Devise a second solution to Problem 3 above by introducing auxiliary segments different from the ones you used in the solution of Problem 3.
5. If AC = AB and CD = BD in the plane figure, show ∠ACD ≅ ∠ABD. Devise a proof that works if the figure does not lie in the plane.



6-5. Betweenness and Separation.

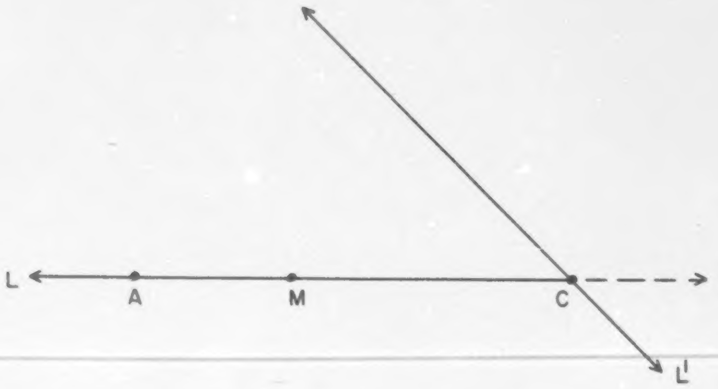
Critical students may have discovered two places in Chapter 5 where the given proofs are not quite complete. These defects occur in Theorems 5-3 and 5-6, and are similar in the two places, consisting of a failure to show why a certain point lies in the interior of a certain angle. In Theorem 5-3 we must know that D is in the interior of $\angle BAC$ before we can conclude that $\overrightarrow{AD}$ bisects this angle. And in steps 9 and 10 of Theorem 5-6 we must know that H is in the interior of $\angle ABC$ and of $\angle AE'C$ before we can apply the Angle Addition Postulate.

In these places it is not enough to observe that in the figure the points lie in the proper places. Remember first that a drawing is only an approximation to the true geometrical situation, and secondly that this is only one figure and the theorem is supposed to be proved for all cases.

You probably wonder why an incomplete proof should be presented in a text-book. The reason is that the proofs of of such separation properties as this one are often long, complicated, and uninteresting, and that they contribute little or nothing to the essential idea of the proof. If you understand the proof of these theorems as given but did not notice the incompleteness of these particular steps, you need not worry about your competence in geometry. For many centuries learned men disputed whether steps like these needed any justification.

However, mathematicians now agree that even such "obvious" steps require a logical proof, and so we present here two theorems and some problems to fill the gaps in these (and later) proofs.

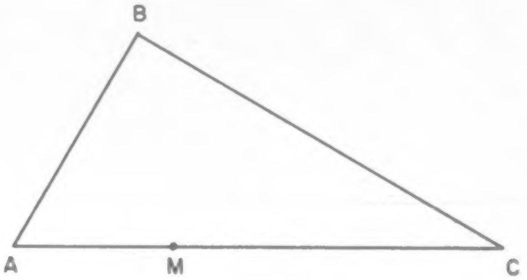
Theorem 6-5. If M is between A and C on a line L, then M and A are on the same side of any other line that contains C.



Proof: The proof will be indirect. If M and A are on opposite sides of L' (in the plane that contains L and L') then some point D of L' lies on the segment $\overline{AM}$. Therefore D is between A and M, by definition of a segment. But D lies on both L and L'. Therefore D = C. Therefore C is between A and M. This is impossible, because M is between A and C. (See Theorem 2-3.)

We can now prove a theorem which completes the proof of Theorems 5-3 and 5-6:

Theorem 6-6. If M is between A and C, and B is any point not on the line $\overleftrightarrow{AC}$, then M is in the interior of $\angle ABC$.

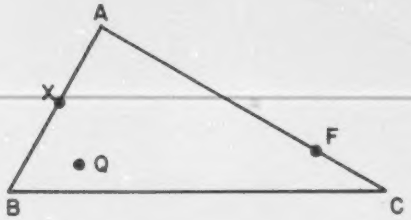


Proof: By the preceding theorem, we know that M and A are on the same side of $\overleftrightarrow{BC}$. By another application of the preceding theorem (interchanging A and C) we know that M and C are on the same side of $\overleftrightarrow{AB}$. By definition of the interior of an angle, these two statements tell us that M is in the interior of ∠ ABC, which was to be proved.

Problem Set 6-5

Note: On this problem set no information is to be read from a figure.

1.



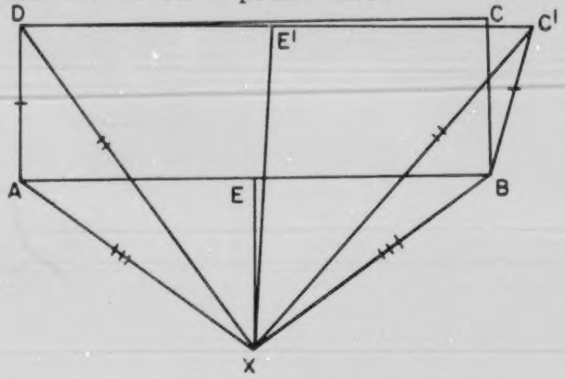
Given Δ ABC with F between A and C, X between A and B and Q in the interior of Δ ABC. Complete the following statements, and give reasons to justify your answers.

a. F lies in the interior of ∠ __________.

b. X lies in the interior of ∠ __________.

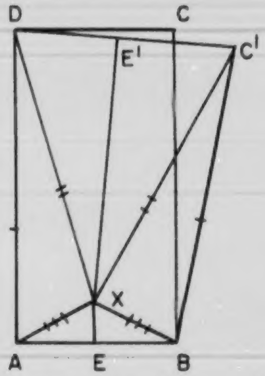
c. Q lies in the interior of ∠ __________, ∠ __________, and ∠ __________.

2. The following faulty argument that an obtuse angle is congruent to a right angle emphasizes the importance of knowing the side of a line on which a point lies.



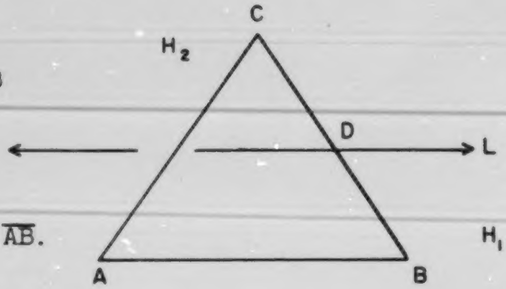
Suppose that ABCD is a rectangle as shown and that the side $\overline{BC}$ is swung outward so that $BC' = BC$ and $\angle ABC'$ is obtuse. Let the perpendicular bisector of $\overline{AB}$ intersect the perpendicular bisector of $\overline{DC'}$ at X. If X is below $\overleftrightarrow{AB}$ as shown, we have $\Delta AXD \cong \Delta BXC'$ by the S.S.S. Theorem, and hence $m\angle DAX = m\angle C'BX$. Also, $\Delta EAX \cong \Delta EBX$ by S.S.S., and so $m\angle EAX = m\angle EBX$. It follows by subtraction that $m\angle DAE = m\angle C'BE$.

In case X lies above $\overleftrightarrow{AB}$, as in the figure below,



we get, exactly as before, $m\angle DAX = m\angle C'BX$, $m\angle EAX = m\angle EBX$, and the desired equality, $m\angle DAE = m\angle C'BE$ follows by addition.

What is wrong with the above argument?

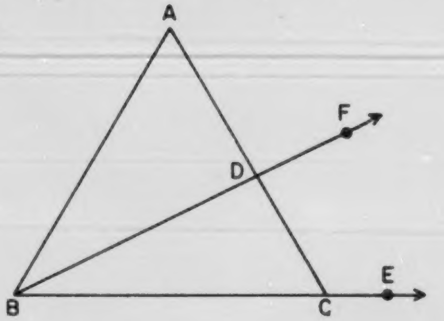


*3. Suppose ABC is a triangle and D is a point between B and C. Show that if L is a line in the plane of Δ ABC which intersects $\overline{BC}$ at D, then L intersects $\overline{AC}$ or $\overline{AB}$. (Hint: If L contains B, then L intersects $\overline{AB}$. If L does not contain B, then let H_1, H_2 be the two half-planes into which L separates the the plane of ΔABC, H_1 being the one that contains B. Since A belongs to either L, H_1, or H_2, there are three cases to consider.)

*4. A theorem whose truth appears obvious is often difficult to prove. The following such theorem is assumed in the proof of Theorem 7-1 of the next chapter.

Suppose ABC is a triangle, D is a point between A and C and E is a point of $\overrightarrow{BC}$ beyond C. Then each point F of $\overrightarrow{BD}$ beyond D is in the interior of $\angle$ACE.

The thing to be proved is that F is on the same side of $\overleftrightarrow{BC}$ as A and that F is on the same side of $\overleftrightarrow{AC}$ as E.



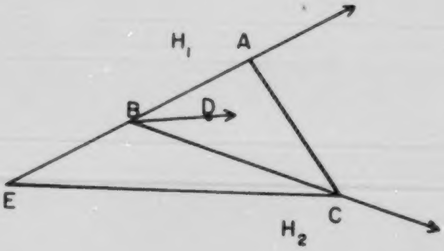
a. How do we know that A and D are on the same side of $\overleftrightarrow{BC}$? What theorem implies that D and F are on this same side?

b. Prove that if H_1, H_2 are the two half planes into which $\overleftrightarrow{AC}$ separates the plane of the figure and B belongs to H_1, then each of E, F belong to H_2. This shows that E and F are on the same side of $\overleftrightarrow{AC}$.

*5. Another theorem whose truth is frequently accepted without proof is the following: If D is a point in the interior of $\angle ABC$, then $\overrightarrow{BD}$ intersects $\overline{AC}$.

We suggest below a "tricky" proof in which we consider $\triangle EAC$, where E is a point of $\overrightarrow{AB}$ beyond B. This enables us to apply the results of Problem 2. Parts a and b below are used to show that $\overleftrightarrow{BD}$ does not intersect $\overline{EC}$.

a. Suppose H_1, H_2 are the two half-planes into which $\overleftrightarrow{BC}$ divides the plane of $\triangle EAC$ with A in H_1. Why is D in H_1? What theorem implies that each point of $\overrightarrow{BD}$ other than B is in H_1? Why is E in H_2?



What theorem implies that each point of $\overline{EC}$ other than C is in H_2? Why does $\overline{EC}$ fail to intersect $\overrightarrow{BD}$?

b. Why does $\overline{EC}$ fail to intersect the ray opposite $\overrightarrow{BD}$?

c. Why does $\overrightarrow{BD}$ intersect $\overline{AC}$?

d. Why does the ray opposite $\overrightarrow{BD}$ fail to intersect $\overline{AC}$?

*6. The following theorem may be used instead of Parts a and b of Problem 5 to show that A and C lie on different sides of $\overleftrightarrow{BD}$.

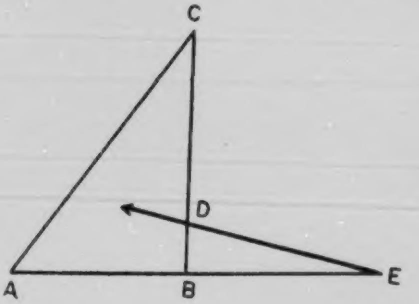
Theorem: If point D is in the interior of $\angle ABC$, then A is not in the interior of $\angle DBC$ nor is C in the interior of $\angle ABD$.

Prove this theorem.

*7. There are studies of geometry that use other systems of postulates than the ones we have adopted. A postulate taken from one such system is the following:

If A, B, C, D, E are points such that A, B and C are non-collinear and B is between A and E and D is between B and C, then there is a point X such that X is between A and C while D is between E and X.

This statement can be proved in our system of postulates.

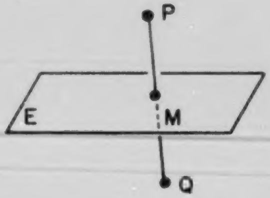


a. Why are A, B, C, D, E coplanar?

b. Show from the Plane Separation Postulate that $\overleftrightarrow{ED}$ intersects $\overline{AC}$ at a point X between A and C.

c. It can be shown that D is between E and X by showing that E and X are on opposite sides of some line. What line?

8. Given points P and Q on opposite sides of plane E with $\overline{PQ}$ intersecting E in M. Identify the following statements as true or false.



a. If L is a line in E perpendicular to $\overleftrightarrow{PQ}$, then P and Q are on opposite sides of L in the plane determined by P and L.

b. If L is a line in E through M, then P and Q are on opposite sides of L in the plane determined by P and L.

c. If L is a line in E, then P and Q are on opposite sides of L in the plane determined by P and L.

d. P and Q are on opposite sides of every plane through M not containing $\overleftrightarrow{PQ}$.

Chapter 7

GEOMETRIC INEQUALITIES

7-1. Making Reasonable Conjectures.

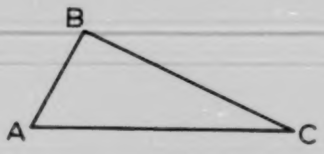
Up to now, in our study of the geometry of the triangle, we have been dealing only with conditions under which we can say that two segments are of equal length, or two angles are of equal measure. We will now proceed to study conditions under which we can say that one segment is longer than another, (that is, has a greater length), or one angle is larger than another, (that is, has a greater measure).

We shall not start, however, by proving theorems. Let us start, rather, by making some reasonable conjectures about the sort of statements that ought to be true. (These statements should not be called theorems unless and until they are proved.)

An example: Given a triangle with two sides of unequal length, what can we say about the angles opposite these sides?

Notice that this problem is naturally suggested by Theorem 5-2, which says that if two sides of a triangle have the same length, then the angles opposite them have the same measure.

You can investigate this situation by sketching a triangle with two sides of obviously unequal lengths, like this:



Here BC is greater than AB, and $m\angle A$ is greater than $m\angle C$. After sketching a few more triangles, you will become pretty well convinced that the following statement ought to be true:

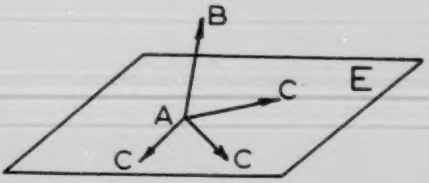
If two sides of a triangle are of unequal length, then the angles opposite them are of unequal measure, and the larger angle is opposite the longer side.

Now try the same sort of procedure with the following problems.

Problem Set 7-1

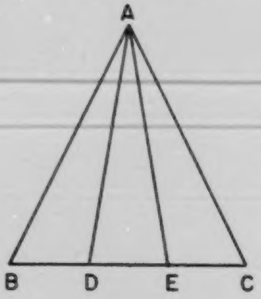
Here are some experiments for you to try.

1. Consider triangles with two angles of unequal measure. Write a statement which you think may be true concerning the sides opposite those angles.
2. Consider several triangles ABC. How does AB + BC compare with AC? BC + AC compare with AB? AB + AC compare with BC? These responses suggest a general conclusion. If you think this conclusion is true for all triangles, write it as a proposition.
3. Consider a quadrilateral RSTQ. How does RS + ST + TQ compare with RQ? Write a proposition suggested by your answer.
4. Draw several triangles in which the measure of one angle is successively greater but the adjacent sides remain unchanged in length. What happens to the length of the third side?
5. Draw $\triangle$ DEF and $\triangle$ XYZ such that DE = XY, FE = ZY, and $m\angle$ DEF $> m\angle$ XYZ. Compare DF and XZ.
6. Regarding $\triangle$ PDQ and $\triangle$ JUN such that $m\angle$PDQ $= m\angle$ JUN, PD > JU, and QD = NU, a hasty person might conclude that PQ > JN. Draw a figure showing that the conclusion is not justified.
7. A is a point in plane E, $\overrightarrow{AB}$ is a ray not lying in E, and $\overrightarrow{AC}$ is a ray lying in E. Considering different positions of $\overrightarrow{AC}$, describe as accurately as you can the position of $\overrightarrow{AC}$ which makes $\angle$ BAC as small as possible; as large as possible. No proof is expected but you are asked to guess the answer on the basis of your knowledge of space.



8. On the basis of drawings decide whether or not an angle can be trisected by the following procedure:

Let Δ ABC be an isosceles triangle with congruent sides $\overline{AB}$ and $\overline{AC}$. Trisect side $\overline{BC}$ with points D, E so that $BD = DE = EC$. Is $\angle BAD \cong \angle DAE \cong \angle EAC$?



7-2. Algebra of Inequalities.

Before considering geometric inequalities we review some of the facts concerning inequalities between real numbers. Note first that $a < b$ and $b > a$ are merely two ways of writing the same thing; we use whichever is more convenient, e.g. $3 < 5$ or $5 > 3$.

Definitions. A real number is positive if it is greater than zero; it is negative if it is less than zero.

We now restate the order postulates, giving examples of their use.

O-1. (Uniqueness of order.) For every x and y, one and only one of the following relations holds: $x < y$, $x = y$, $x > y$.

O-2. (Transitivity of order.) If $x < y$ and $y < z$, then $x < z$.

Example 1. $3 < 5$ and $5 < 9$, hence, $3 < 9$.

Example 2. If we know that $a < 3$ and $b > 3$, we can conclude that $a < b$. Proof: If $a < 3$ and $3 < b$, then $a < b$.

Example 3. Any positive number is greater than any negative number.

Given: p is positive, n is negative.

To prove: $p > n$.

Proof:

1. p is positive.	1. Given.
2. $p > 0$.	2. Definition of positive.

3. $0 < p$.	3. Relation between $<$ and $>$.
4. $n < 0$.	4. Definition of negative.
5. $n < p$.	5. Postulate 0-2.
6. $p > n$.	6. Relation between $<$ and $>$.

0-3. (Addition for inequalities.) If $x < y$, then $x + z < y + z$, for every z.

Example 4. Since $3 < 5$ it follows that $3 + 2 < 5 + 2$, or $5 < 7$; that $3 + (-3) < 5 + (-3)$, or $0 < 2$; that $3 + (-8) < 5 + (-8)$, or $-5 < -3$.

Example 5. If $a < b$ then $-b < -a$. Proof: $a + (-a-b) < b + (-a-b)$, or $-b < -a$.

Example 6. If $a + b = c$ and b is positive, then $a < c$.

Proof:

1. b is positive.	1. Why?
2. $b > 0$.	2. Why?
3. $0 < b$.	3. Why?
4. $a < a + b$.	4. Why?
5. $a < c$.	5. Why?

Example 7. If $a + b < c$ then $a < c - b$. Proof left to the student.

Example 8. If $a < b$, then $c - a > c - b$ for every c. Proof left to the student.

0-4. (Multiplication for Inequalities.) If $x < y$ and $z > 0$, then $xz < yz$.

Example 9. From $3 < 6$ we can conclude that $3000 < 6000$; also, that $\frac{1}{18} \cdot 3 < \frac{1}{18} \cdot 6$, or $\frac{1}{6} < \frac{1}{3}$.

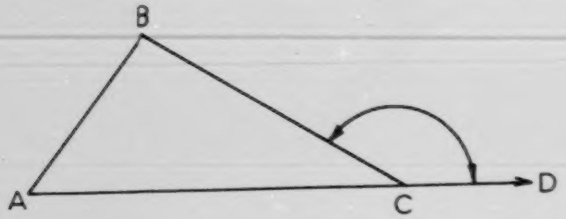
Example 10. If $x < y$ and $z < 0$, then $xz > yz$. Proof left to the student.

0-5. (Addition of Inequalities.) If $a < b$ and $x < y$, then $a + x < b + y$.

This is not a postulate but a theorem; its proof is given in Section 2-2. However, it is convenient to list it, for reference, along with the postulates.

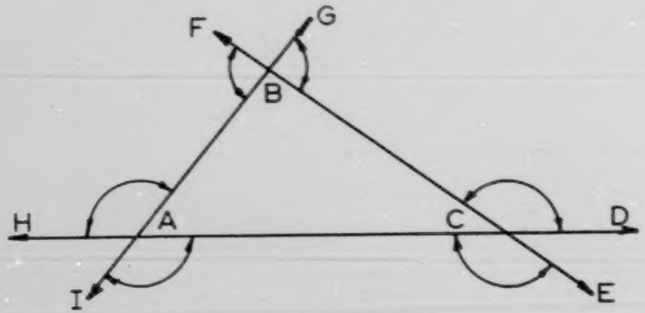
7-3. The Basic Inequality Theorems.

In the figure below, the angle $\angle$ BCD is called an exterior angle of Δ ABC. More precisely:



Definition. If C is between A and D, then $\angle$ BCD is an exterior angle of Δ ABC.

Every triangle has six exterior angles, as indicated by the double-headed arrows in the figure below:



These six angles form three pairs of congruent angles, because they form three pairs of vertical angles.

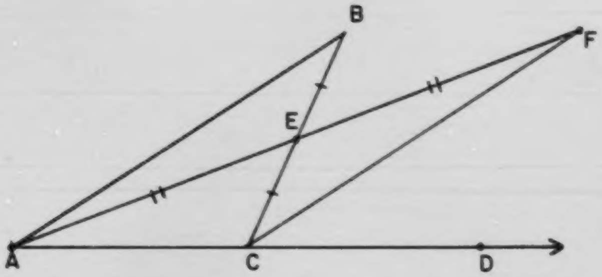
Definition. $\angle$A and $\angle$B of the triangles are called the remote interior angles of the exterior angles $\angle$ BCD and $\angle$ ACE.

Similarly, $\angle$ A and $\angle$ C of Δ ABC are the remote interior angles of the exterior angles $\angle$ ABF and $\angle$ CBG.

Theorem 7-1. (The Exterior Angle Theorem.) An exterior angle of a triangle is larger than either remote interior angle.

Restatement: Note first that the two exterior angles at vertex C, above, have equal measures (vertical angles), and so it doesn't matter which of them we compare with $\angle A$ and $\angle B$. It turns out to be easiest to compare $m\angle BCD$ with $m\angle B$ and $m\angle ACE$ with $m\angle A$. Since the proofs of these two cases are exactly similar we need prove only one.

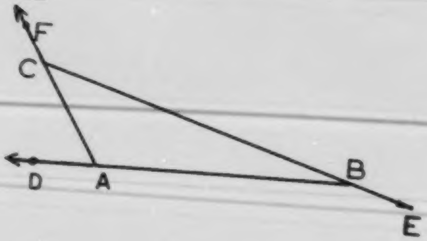
Given triangle $\triangle ABC$. If C is between A and D, then $m\angle BCD > m\angle B$.



Proof:

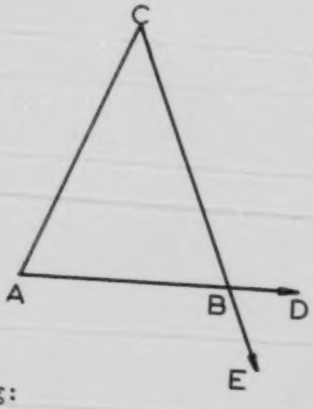
Statements	Reasons
1. Let E be the mid-point of $\overline{BC}$.	1. By Theorem 2-5 there is such a mid-point.
2. Let F be a point of the ray opposite to $\overrightarrow{EA}$, such that $EF = EA$.	2. By Theorem 2-4, there is such a point.
3. $\angle BEA \cong \angle FEC$.	3. Vertical angles are congruent.
4. $\triangle BEA \cong \triangle CEF$.	4. Statements 1, 2, 3 and the S.A.S. Postulate.
5. $m\angle B = m\angle ECF$.	5. Corresponding parts of congruent triangles.
6. $m\angle BCD = m\angle ECF + m\angle FCD$.	6. Postulate 13 (The Angle Addition Postulate.)
7. $m\angle BCD = m\angle B + m\angle FCD$.	7. Statements 5 and 6.
8. $m\angle BCD > m\angle B$.	8. By algebra from step 7. (Since $m\angle FCD$ is a positive number, Example 6 of Section 7-2 applies.)

Problem Set 7-3a



1. a. Name the remote interior angles of the exterior angle $\angle ABE$ in the figure.

   b. $\angle ABC$ and $\angle BAC$ are the remote interior angles of which exterior angle?



2. a. In the figure, which angles are exterior angles of the triangle?

   b. What is the relationship of $m\angle DBC$ to $m\angle A$? Why?

   c. What is the relationship of $m\angle DBC$ to $m\angle C$? Why?

   d. What is the relationship of $m\angle DBC$ to $m\angle CBA$? Why?

3. Using the figure, complete the following:

   a. If $x = 40$ and $y = 30$, then $m\angle BCE >$ _____.

   b. If $x = 72$ and $y = 73$, then $m\angle BCE$ _____.

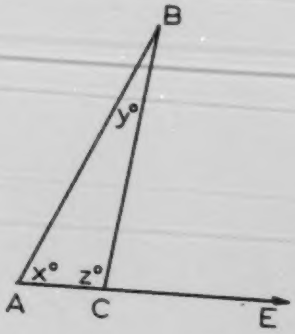
   c. If $y = 54$ and $z = 68$, then $m\angle BCE$ _____.

   d. If $m\angle BCE = 112$, then x_____.

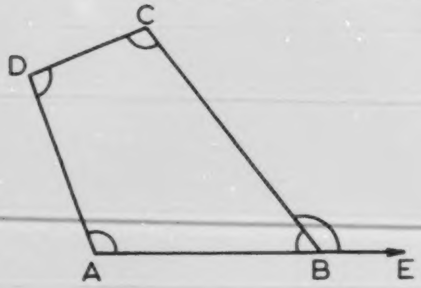
   e. If $m\angle BCE = 150$, then z_____.

   f. If $x = 25$ and $z = 90$, then $m\angle BCE$ _____.

   g. If $x = 90$ and $y = 90$, then $m\angle BCE$ _____.



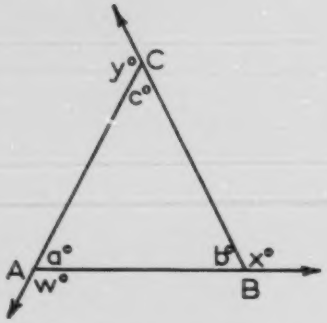
4. The accompanying figure is an illustration of this statement: An exterior angle of a quadrilateral is greater than each of the remote interior angles. Is this a true statement? Explain.



*5. Prove the following theorem: The sum of the measures of any two angles of a triangle is less than 180.

Given: ΔABC with angle measures as in the figure.

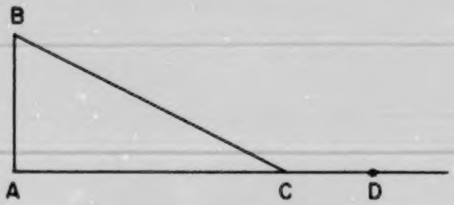
Prove: $a + b < 180$.
$b + c < 180$.
$a + c < 180$.



*6. Prove the following theorem: The base angles of an isosceles triangle are acute. (Hint: Base your proof on the statement of the previous problem.)

Theorem 7-1, while perhaps not very exciting in itself, is extremely useful in proving other theorems. (A theorem of this type is sometimes called a lemma.) For example, the following is a useful corollary.

Corollary 7-1-1. If a triangle has a right angle, then the other two angles are acute.



Proof: If $m\angle A = 90$, then $m\angle BCD > 90$, and therefore, $m\angle BCA < 90$. In a similar way we can prove $m\angle ABC < 90$.

We next use Theorem 7-1 to prove two more congruence theorems.

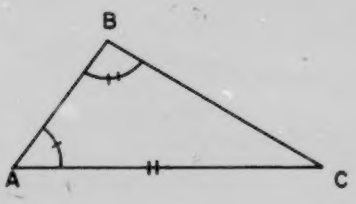
Theorem 7-2. (The S.A.A. Theorem) Given a correspondence between two triangles. If two angles and a side opposite one of them in one triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

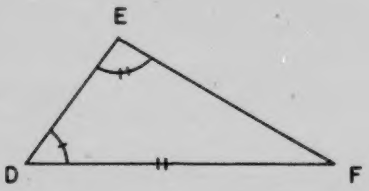
Restatement: Let $ABC \leftrightarrow DEF$ be a correspondence between two triangles. If

$$\angle A \cong \angle D,$$
$$\angle B \cong \angle E,$$

and $\overline{AC} \cong \overline{DF}$,

then $\triangle ABC \cong \triangle DEF$.



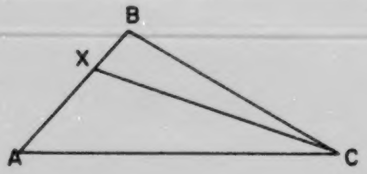


Proof:

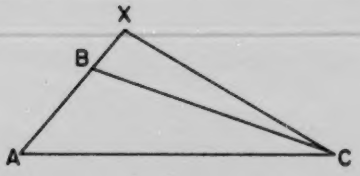
Statements	Reasons
1. On $\overrightarrow{AB}$ take X so that $AX = DE$.	1. Point Plotting Theorem.
2. $\triangle AXC \cong \triangle DEF$.	2. S.A.S. Postulate.
3. $m\angle AXC = m\angle DEF$.	3. Definition of congruence.
4. $m\angle AXC = m\angle ABC$.	4. Step 3 and given.

Now suppose that X is not the same point as B.

5. Either X is between A and B or B is between A and X.	5. Step 1 and definition of ray.



OR



6. In either case one of $\angle AXC$ and $\angle ABC$ is an exterior angle of ΔBXC and the other is a remote interior angle.	6. Definition of exterior angle and remote interior angle.
7. $m\angle AXC \neq m\angle ABC$.	7. Step 6 and Theorem 7-1.
8. $X = B$.	8. Step 7 contradicts Step 4.
9. $\Delta ABC \cong \Delta DEF$.	9. Steps 2 and 8.

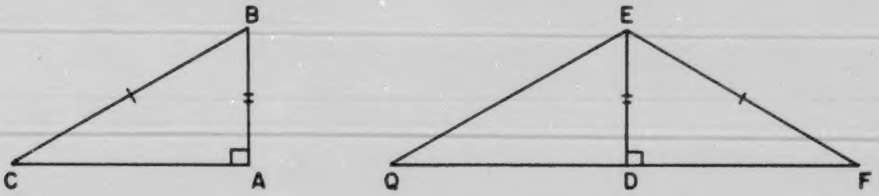
Although it was pointed out in connection with the S.A.S. Postulate that an S.S.A. theorem cannot in general be proved, there is one special case; namely, the case in which the angle is a right angle, that follows from Theorem 7-2.

Theorem 7-3. (The Hypotenuse-Leg Theorem.) Given a correspondence between two right triangles. If the hypotenuse and one leg of one triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

Restatement: In ΔABC and ΔDEF let $m\angle A = m\angle D = 90$. Let $ABC \longleftrightarrow DEF$ be a correspondence such that

$$BC = EF \text{ and } AB = DE.$$

Then $\Delta ABC \cong \Delta DEF$.



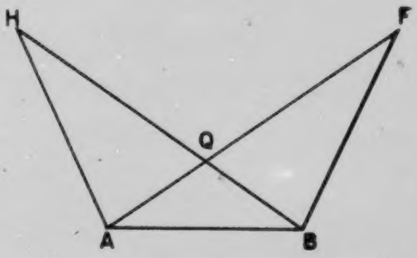
Proof: On the ray opposite to $\overrightarrow{DF}$ take Q such that DQ = AC. Then Δ DEQ $\cong \Delta$ ABC by the S.A.S. Postulate, and so EQ = BC. Δ EQF is thus an isosceles triangle, and so $\angle$ EQD $\cong \angle$ EFD. In Δ DEQ and Δ DEF we thus have

$$\overline{EQ} \cong \overline{EF},\ \angle\ EQD \cong \angle\ EFD \text{ and } \angle\ EDQ \cong \angle\ EDF.$$

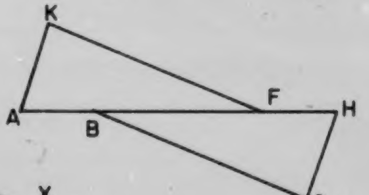
Hence, by the S.A.A. Theorem, Δ DEF $\cong \Delta$ DEQ. Since we have already established Δ DEQ $\cong \Delta$ ABC we conclude that Δ DEF $\cong \Delta$ ABC, which is what we wanted.

Problem Set 7-3b

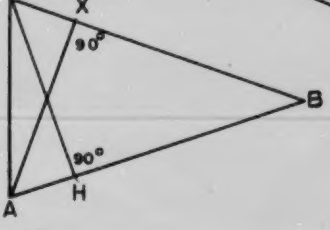
1. If in this figure AQ = BQ and $\angle$H $\cong \angle$ F, prove that FB = HA.



2. Given that $\overline{AK} \perp \overline{KF}$, $\overline{HQ} \perp \overline{QB}$, AB = HF, AK = HQ. Prove that KF = QB.

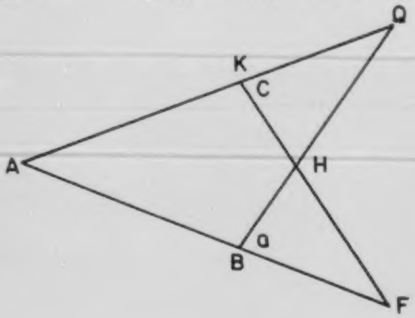


3. If AX = FH in this figure, prove that FB = AB.

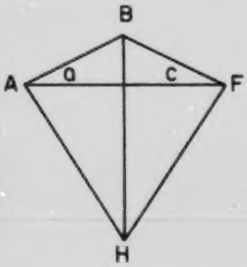


4. If two altitudes of a triangle are congruent, the triangle is isosceles.

5. In this figure: $\angle c \cong \angle a$.
$AQ = AF$.
Prove: $QB = FK$.

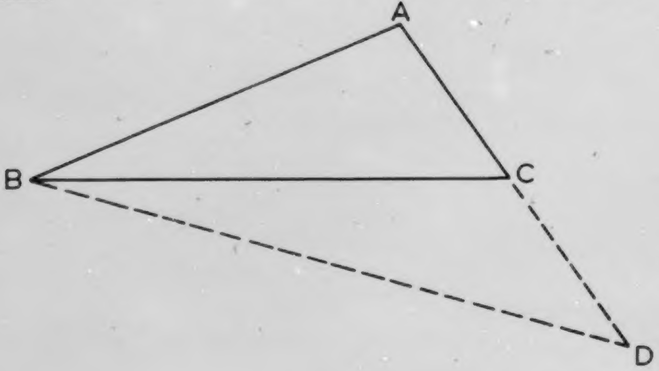


6. In this figure if $\angle a \cong \angle c$, $\overline{AB} \perp \overline{AH}$ and $\overline{FB} \perp \overline{FH}$, prove that $AH = FH$.



Theorem 7-4. If two sides of a triangle are not congruent, then the angles opposite these two sides are not congruent, and the larger angle is opposite the longer side.

Restatement: Given $\triangle ABC$. If $AB > AC$, then $m\angle C > m\angle B$.



Proof: Let D be a point of $\overrightarrow{AC}$, such that $AD = AB$. (By the Point Plotting Theorem, there is such a point.) Since the base angles of an isosceles triangle are congruent, we have

(1) $$m\angle ABD = m\angle D.$$

Now $AD > AC$, since $AD = AB$ and $AB > AC$, and so C is between A and D by Theorem 2-1. By Theorem 6-6, C is in the interior of ABD, and so

(2) $$m\angle ABD = m\angle ABC + m\angle CBD$$

by the Angle Addition Postulate. Since $m\angle CBD > 0$ it follows that

(3) $$m\angle ABD > m\angle ABC.$$

Therefore

(4) $$m\angle D > m\angle ABC, \text{ from (1) and (3).}$$

Since $\angle ACB$ is an exterior angle of ΔBCD, we have

(5) $$m\angle ACB > m\angle D.$$

By (4) and (5),

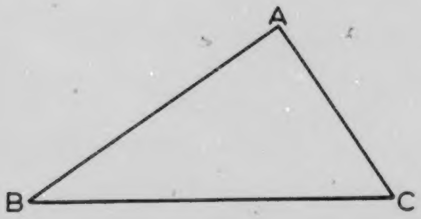
$$m\angle ACB > m\angle ABC,$$

that is,

$$m\angle C > m\angle B,$$

which was to be proved.

Theorem 7-5. If two angles of a triangle are not congruent, then the sides opposite them are not congruent, and the longer side is opposite the larger angle.



Restatement: In any triangle ΔABC, if $m\angle C > m\angle B$, then $AB > AC$.

Proof: We want to prove that $AB > AC$. Since AB and AC are numbers, there are only three possibilities: (1) $AB = AC$, (2) $AB < AC$ and (3) $AB > AC$. The method of the proof is to show that the first two of these "possibilities" are in fact impossible. The only remaining possibility will be (3), and this will mean that the theorem is true.

(1) If $AB = AC$, then by Theorem 5-2 it follows that $\angle B \cong \angle C$; and this is false. Therefore, it is impossible that $AB = AC$.

(2) If $AB < AC$, then by Theorem 7-4 it follows that $m\angle C < m\angle B$; and this is false. Therefore, it is impossible that $AB < AC$.

The only remaining possibility is that $AB > AC$, which was to be proved.

The proof of Theorem 7-5, as we have given it, is merely a handy way of stating an indirect proof. It could have been written more formally, like this:

"Suppose that Theorem 7-5 is false. Then either $AB = AC$ or $AB < AC$. It is impossible that $AB = AC$, because . . . . And it is impossible that $AB < AC$, because . . . . Therefore, 7-5 is not false."

The proof is probably easier to read, however, the way we gave it the first time. We will be using the same sort of scheme again. That is, we will list the possibilities, in a given situation, and then show that all but one of these "possibilities" are in fact impossible; it will then follow that the last remaining possibility must represent what actually happens.

"That process starts upon the supposition that when you have eliminated all which is impossible, then whatever remains, however improbable, must be the truth." (Sherlock Holmes in "The Adventure of the Blanched Soldier".)

Theorems 7-4 and 7-5 are related in a special way; they are called converses of one another. To get one from the other, we interchange the hypothesis and the conclusion. We can exhibit this fact by restating the theorems this way:

Theorem 7-4. Given $\triangle ABC$. If $AB > AC$, then $m\angle C > m\angle B$.

Theorem 7-5. Given $\triangle ABC$. If $m\angle C > m\angle B$, then $AB > AC$.

We have seen lots of pairs of theorems that are related this way. For example, we showed that if a triangle is isosceles, then its base angles are congruent; and later we showed that if the base angles of a triangle are congruent, then the triangle is isosceles. Each of these theorems is the converse of the other. We showed that every equilateral triangle is equiangular; and later we proved the converse, which states that every equiangular triangle is equilateral.

It is very important to remember that the converse of a true theorem is not necessarily true at all. For example, the theorem "vertical angles are congruent" is always true, but the converse, "congruent angles are vertical" is certainly not true in all cases. If two triangles are congruent, then they have the same area, but if two triangles have the same area, it does not follow that they are congruent. If $x = y$, then it follows that $x^2 = y^2$; but if $x^2 = y^2$, it does not follow that $x = y$. (The other possibility is that $x = -y$.) It is true that every physicist is a scientist, but it is not true that every scientist is a physicist.

If a theorem and its converse are both true, they can be conveniently combined into a single statement by using the phrase "if and only if". Thus, if we say:

> Two angles of a triangle are congruent if and only if the opposite sides are congruent;

we are including in one statement both theorems on isosceles triangles. The first half of this double statement:

> Two angles of a triangle are congruent if the opposite sides are congruent;

is Theorem 5-2; and the second half:

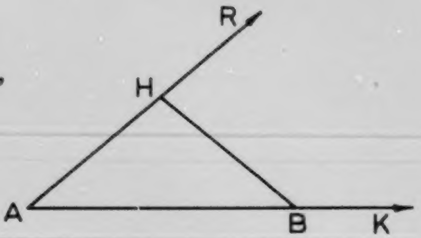
> Two angles of a triangle are congruent only if the opposite sides are congruent;

is a restatement of Theorem 5-5.

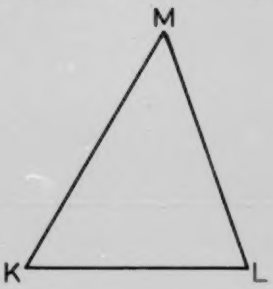
Problem Set 7-3c

1. In $\triangle GHK$, $GH = 5$, $HK = 14$, $KG = 11$. Name the largest angle. Name the smallest angle.
2. In $\triangle ABC$, $m\angle A = 36$, $m\angle B = 74$, and $m\angle C = 70$. Name the longest side. Name the shortest side.

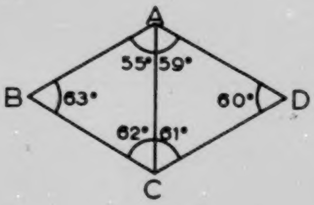
3. Given the figure with $HA = HB$, $m\angle HBK = 140$, and $m\angle AHB = 100$, fill in the blanks below:
   a. $m\angle A =$ ___.
   b. $m\angle RHB =$ ___.
   c. ___ is the longest side of $\triangle ABH$.



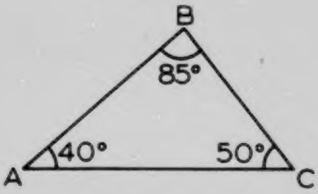
4. What conclusion can you reach about the length of $\overline{ML}$ in $\triangle KLM$ if:
   a. $m\angle K > m\angle M$?
   b. $m\angle K < m\angle L$?
   c. $m\angle M > m\angle K > m\angle L$?
   d. $m\angle M > m\angle L$?
   e. $m\angle K > m\angle M$ and $m\angle K > m\angle L$?
   f. $m\angle K \geq m\angle L$ and $m\angle M \leq m\angle L$?



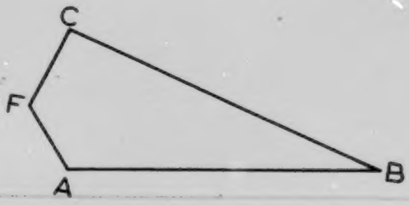
5. If the figure were correctly drawn which segment would be the longest?



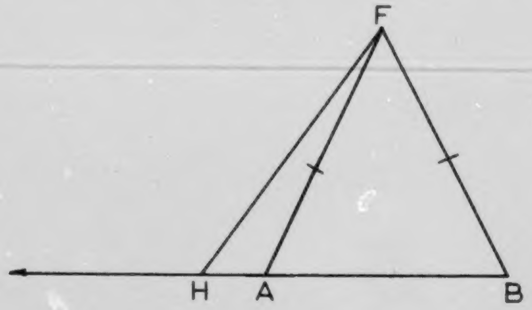
6. Name the sides of the figure in order of increasing length.



7. If in the figure $\overline{AF}$ is the shortest side and $\overline{CB}$ is the longest side, prove that $m\angle F > m\angle B$. (Hint: use diagonal $\overline{FB}$.)



*8. If the base of an isosceles triangle is extended, a segment which joins the vertex of the triangle with any point in this extension is greater than one of the congruent sides of the triangle.

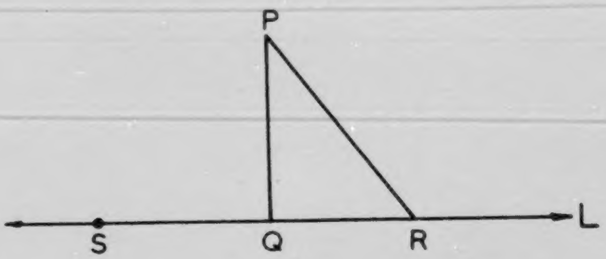


9. Write the converse of each statement. Try to decide whether each statement, and each converse, is true or false.
   a. If a team has some spirit, it can win some games.
   b. If two angles are right angles, they are congruent.
   c. Any two congruent angles are supplementary.
   d. The interior of an angle is the intersection of two half-planes.
   e. If Joe has scarlet fever, he is seriously ill.
   f. If a man lives in Cleveland, Ohio, he lives in Ohio.
   g. If the three angles of one triangle are congruent to the corresponding angles of another triangle, the triangles are congruent.
   h. If two angles are complementary, the sum of their measures is 90.

10. When asked to give the converse of this statement, "If I hold a lighted match too long, I will be burned", John said, "I will be burned if I hold a lighted match too long." Was John's sentence the converse of the original statement? Discuss.

11. a. Is a converse of a true statement always true? Which parts of Problem 9 illustrate your answer?
    b. May a converse of a false statement be true? Which parts of Problem 9 illustrate your answer?

Theorem 7-6. The shortest segment joining a point to a line is the perpendicular segment.



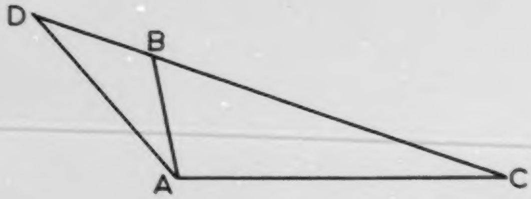
Restatement: Let Q be the foot of the perpendicular to the line L through the point P, and let R be any other point on L. Then $PQ < PR$.

Proof: Let S be a point of L, such that Q is between S and R. Then $\angle PQS$ is an exterior angle of ΔPQR. Therefore, $m\angle PQS > m\angle PRQ$. But $m\angle PQS = m\angle PQR = 90$, and so $m\angle PQR > m\angle PRQ$. By Theorem 7-5 it follows that $PQ < PR$, which was to be proved.

Definition. The distance between a line and a point not on it is the length of the perpendicular segment from the point to the line. The distance between a line and a point on the line is defined to be zero.

Theorem 7-7. (The Triangle Inequality.) The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

Restatement: In any triangle ΔABC, we have $AB + BC > AC$.



Proof: Let D be a point of the ray opposite to $\overrightarrow{BC}$ such that $DB = AB$. Since B is between C and D

$$DC = DB + BC.$$

Then (1) $DC = AB + BC$.

Also (2) $m\angle DAB < m\angle DAC$,

because B is in the interior of $\angle DAC$.

Since ΔDAB is isosceles, with $AB = DB$, it follows that

$$(3)\ m\angle ADB = m\angle DAB.$$

By (2) and (3) we have

$$m\angle ADB < m\angle DAC.$$

Applying Theorem 7-5 to ΔADC, we see that

$$(4)\ DC > AC.$$

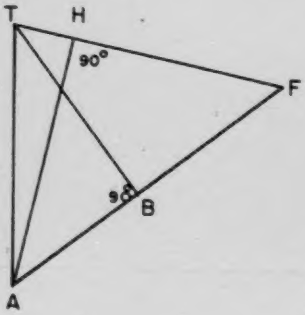
By (1) and (4) it follows that

$$AB + BC > AC,$$

which was to be proved.

Problem Set 7-3d

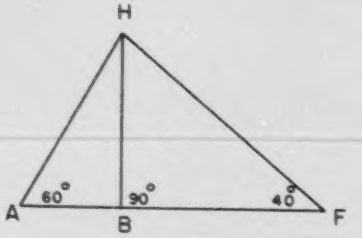
1. Here $AH <$ ____ and $AH <$ ____. $BT <$ ____ and $BT <$ ____. State the theorem involved.



2. With angle measures as shown in the figure, insert HA, HF, HB below in correct order.

   _____ < _____ < _____.

   State theorems to support your conclusion.



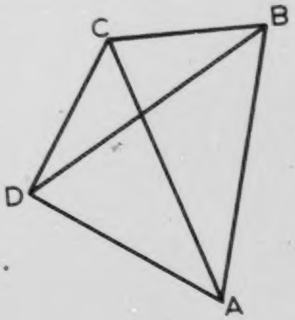
3. Suppose that you wish to draw a triangle with 5 as the length of one side and 8 as the length of a second side. Your third side must have a length greater than ____, and less than ____.

4. Suppose that you wish to draw a triangle with j as the length of one side and k as the length of a second side. It is known that $j < k$. Indicate, as efficiently as you can, the restrictions on the length, x, of the third side.

5. Prove that the sum of the lengths of the diagonals of this quadrilateral is less than the sum of the lengths of its sides.

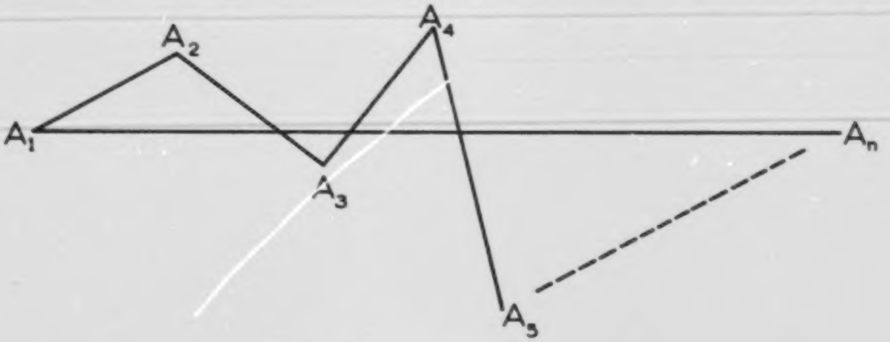
   Given: Quadrilateral ABCD.

   To prove: $DB + CA < AB + BC + CD + DA$.



*6. Let A, B, C, be points, not necessarily different. Prove that $AB + BC \geq AC$ and that $AB + BC = AC$ if and only if B is on the segment $\overline{AC}$.

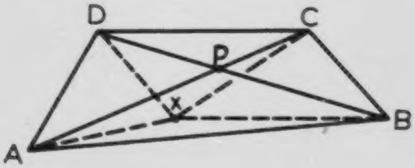
*7. Prove that the shortest polygonal path from one point to another is the segment joining them.



Given: n points A_1, A_2, ..., A_n.

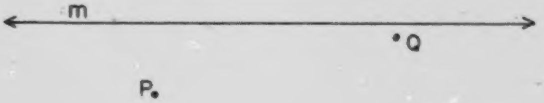
Prove: $A_1A_2 + A_2A_3 + \ldots\ldots + A_{n-1}A_n \geq A_1A_n$.

*8. Given two segments $\overline{AC}$ and $\overline{BD}$ intersecting at P.



Prove that if X is any point in the plane of ABCD other than P, then XA + XB + XC + XD > PA + PB + PC + PD. Will this result be true if X is not in the plane of ABCD?

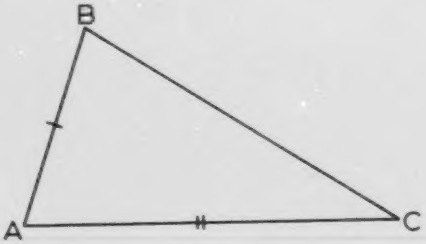
*9. Given a line m and two points P, Q on the same side of m. Find the point R on m for which PR + RQ is as small as possible.

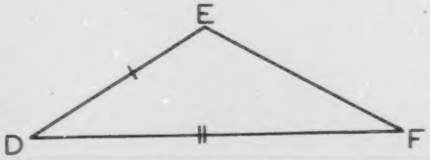


We will now prove a theorem which is a little like Theorem 7-5, except that it deals with two triangles instead of one.

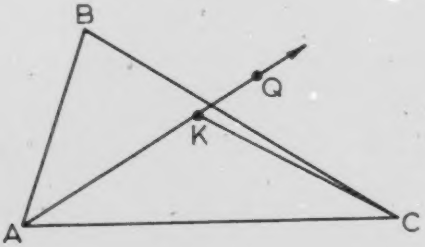
Theorem 7-8. If two sides of one triangle are congruent respectively to two sides of a second triangle, and the included angle of the first triangle is larger than the included angle of the second, then the opposite side of the first triangle is longer than the opposite side of the second.

Restatement: Given $\triangle ABC$ and $\triangle DEF$. If $AB = DE$, $AC = DF$ and $m\angle A > m\angle D$, then $BC > EF$.



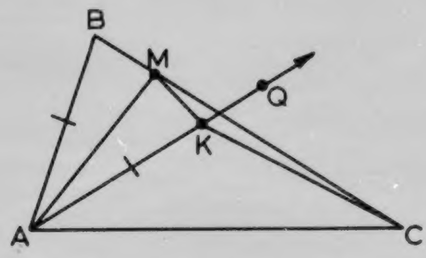


Proof: Step 1. We construct $\triangle AKC$, with K in the interior of $\angle BAC$, such that $\triangle AKC \cong \triangle DEF$, like this:



To do this, we use the Angle Construction Postulate, to get a ray $\overrightarrow{AQ}$, with Q on the same side of $\overleftrightarrow{AC}$ as B such that $\angle QAC \cong \angle D$. On $\overrightarrow{AQ}$ we take a point K such that $AK = DE$. By the S.A.S. Postulate, we now have $\triangle AKC \cong \triangle DEF$, which is what we wanted.

Step 2. Now we bisect $\angle BAK$, and let M be the point where the bisector crosses $\overline{BC}$, like this:



The marks on the figure indicate that $AK = AB$, and this is true, because $AK = DE$ and $DE = AB$.

We are now almost done. By the S.A.S. Postulate, we have $\triangle ABM \cong \triangle AKM$. Therefore, $MB = MK$. By Theorem 7-7, we know that

$$CK < CM + MK.$$

Therefore,

$$CK < CM + MB,$$

because $MB = MK$. Since $CK = EF$ and $CM + MB = BC$, we get $EF < BC$, which is what we wanted.

The converse of this theorem is also true.

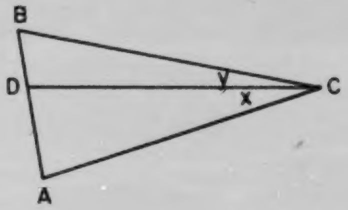
Theorem 7-9. If two sides of one triangle are congruent respectively to two sides of a second triangle, and the third side of the first triangle is longer than the third side of the second, then the included angle of the first triangle is larger than the included angle of the second.

The proof is similar to that of Theorem 7-5, use being made of Theorem 7-6 and the S.S.S. Theorem to eliminate the two unwanted cases. The student should fill in the details.

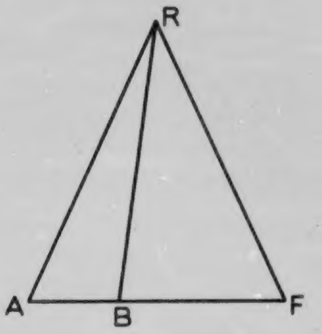
Problem Set 7-3e

1. State the combination of Theorems 7-8 and 7-9 in the "if and only if" form.

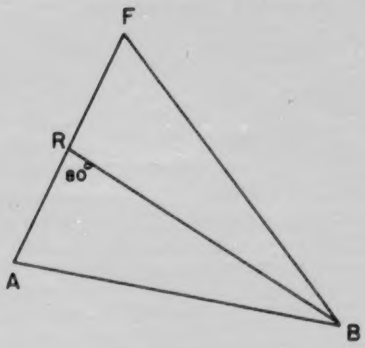
2. In this figure $AC = BC$, and $BD < AD$. Prove: $m\angle x > m\angle y$.



3. In isosceles triangle RAF with $RA = RF$ and B a point on $\overline{AF}$ such that $m\angle ARB < m\angle BRF$. Prove: $AB < BF$.

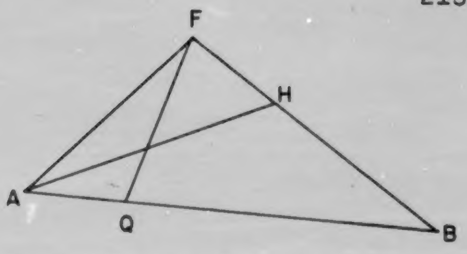


4. Given ΔABF with median $\overline{RB}$ and $m\angle ARB = 80$. Prove: $m\angle A > m\angle F$.



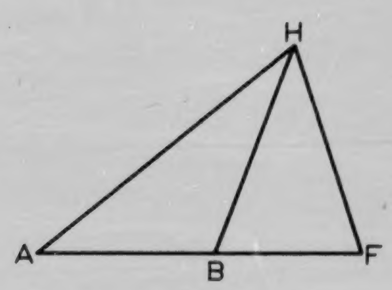
5. In ΔABC, $BC > AC$ and Q is the midpoint of $\overline{AB}$. Is $\angle CQA$ obtuse or acute? Explain.

6. In this figure $FH = AQ$.
$AH > FQ$.
Prove: $AB > FB$.

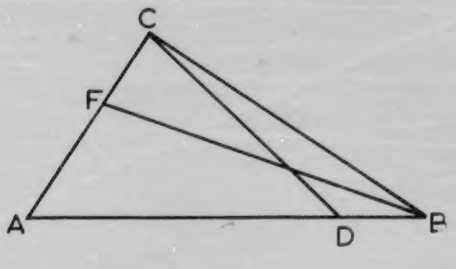


7. A non-equilateral quadrilateral has two pairs of congruent adjacent sides. Prove that the measure of the angle included between the smaller sides is greater than the measure of the angle between the larger sides.

8. Prove the following theorem: If a median of a triangle is not perpendicular to the side to which it is drawn, then the lengths of the other two sides of the triangle are unequal.

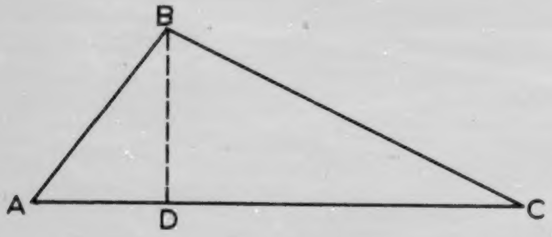


9. Given $AB > AC$ and $FC = DB$ in this figure. Prove that $FB > CD$.



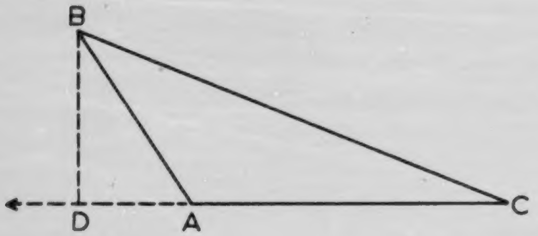
7-4. Altitudes.

Definition. An altitude of a triangle is the perpendicular segment joining a vertex of the triangle to the line that contains the opposite side.

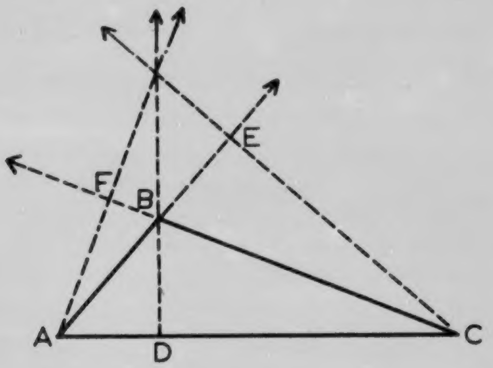


In the figure $\overline{BD}$ is called the altitude from B to $\overleftrightarrow{AC}$, or simply the altitude from B. (Notice that we say the altitude from B instead of an altitude from B, because Theorem 6-3 tells us that there is only one.)

Notice that the foot of the perpendicular does not necessarily lie on the side $\overline{AC}$ of the triangle. The figure may look like this:



Notice also that every triangle has three altitudes, one from each of the three vertices, like this:



Here $\overline{AF}$ is the altitude from A, $\overline{BD}$ is the altitude from B and $\overline{CE}$ is the altitude from C.

It is customary to use the same word "altitude" for two other different, but related, concepts.

(1) The number which is the length of the perpendicular segment is called altitude; thus one may say "The altitude from B is 6", meaning that $BD = 6$.

(2) The line containing the perpendicular segment is also called altitude; a property of the above figure can be expressed by saying that the three altitudes of the triangle intersect in one point. (This property is true for all triangles and will be proved in Chapter 14.)

This triple use of the one word could cause trouble but generally does not, since it is usually easy to tell in any particular case which usage is being made.

Problem Set 7-4

1. Define: a. Altitude of a triangle.
   b. Median of a triangle.
2. Draw an obtuse triangle (a triangle having an obtuse angle) and its three altitudes.

3. In an equilateral triangle a median and an altitude are drawn to the same side. Compare the lengths of these two segments.

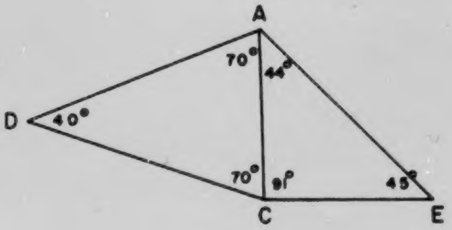
4. Prove that the perimeter of a triangle is greater than the sum of the three altitudes.

5. Prove the following theorem: The altitudes of an equilateral triangle are congruent.

Review Problems

1. Three guy wires of equal length are being used to support a newly planted tree on level ground. If they are all fastened to the tree at the same height on the tree, will they be pegged to the ground at equal distances from the foot of the tree? Why?

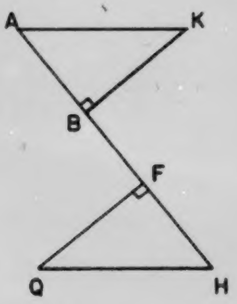
2. If this figure were drawn correctly, which segment in the figure would be the shortest? Explain your reasoning.



3. Prove the following theorem:

   If two oblique (not perpendicular) line segments are drawn to a line from a point on a perpendicular to that line, the one containing the point more remote from the foot of the perpendicular is the longer.

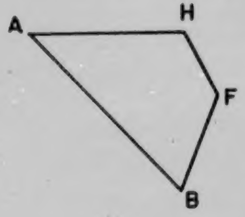
4. In this planar figure,
   $AK = HQ$, $AF = HB$, $\overline{KB} \perp \overline{AH}$, $\overline{QF} \perp \overline{AH}$.
   Prove $\angle Q \cong \angle K$.
   Does $\overline{KQ}$ bisect $\overline{BF}$?



5. In $\triangle ABC$, $AC > AB$. Prove that any line segment from A to a point on $\overline{BC}$ between B and C is shorter than $\overline{AC}$.

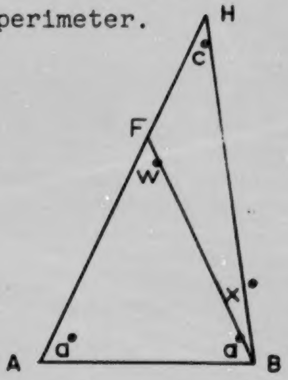
6. Segments drawn from a point in the interior of a triangle to the three vertices have lengths r, s, t. Prove that $r + s + t$ is greater than half the perimeter of the triangle.

7. In this planar figure $\overline{FH}$ is the shortest side and $\overline{AB}$ is the longest side. Prove $m\angle F > m\angle A$.



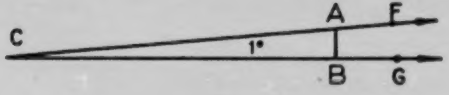
8. Prove the following theorem: The length of the longest side of the triangle is less than half its perimeter.

*9. Given isosceles $\triangle ABF$ with $FA = FB$, $AB < AF$, and H on $\overleftrightarrow{AF}$, so that F is between A and H. Prove no two sides of $\triangle ABH$ are equal in length.



*10. On the basis of the assumptions we have accepted and the theorems we have proved in this course we are not able at present to prove that the sum of the measures of the three angles of a triangle is 180 (an idea with which you have been familiar for some time). But, we can easily construct a triangle and prove that the sum of the measures of the angles of this triangle is less than 181.

Let $\angle FCG$ have measure 1 (Angle Construction Postulate). On $\overrightarrow{CF}$ and $\overrightarrow{CG}$ take points A and B so that $CA = CB$ (Point Plotting Theorem).

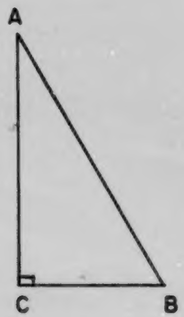


Why is the sum of the measures of the angles of this triangle less than 181?

*11. The sum of the measures of the three angles of a triangle is less than 270.

*12. In this figure:
$\angle C$ is a right angle.
$m\angle B = 2m\angle A$.

Prove: $AB = 2\ CB$.
(Hint: Introduce auxiliary segments.)



*13. Prove this theorem: The sum of the distances from a point within a triangle to the ends of one side is less than the sum of the lengths of the other two sides.

*14. Suppose $\overleftrightarrow{AC}$ intersects $\overleftrightarrow{BD}$ at a point B between A and C. Perpendiculars are dropped from A and C to $\overleftrightarrow{BD}$ striking it at P and Q respectively. Show that P and Q are not on the same side of B.

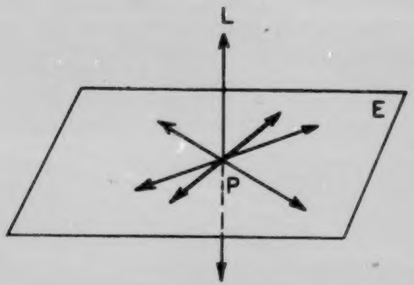
Chapter 8

PERPENDICULAR LINES AND PLANES IN SPACE

8-1. The Basic Definition.

In this chapter we shall be specifically concerned with properties of figures that do not lie in a single plane. The fundamental properties of such figures are stated in Postulates 5b, 6, 7, 8 and 10, and in Theorems 3-2, 3-3 and 3-4. It would be worth your while to review these.

Definition. A line and a plane are perpendicular if they intersect and if every line lying in the plane and passing through the point of intersection is perpendicular to the given line.

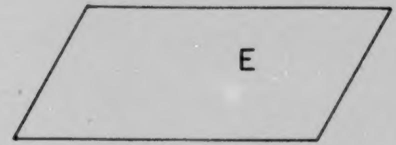


If line L and plane E are perpendicular we write $L \perp E$ or $E \perp L$.

We have indicated, in the figure, three lines in E passing through P. Notice that in a perspective drawing, perpendicular lines don't necessarily look perpendicular. Notice also that if we merely required that E contain one line through P perpendicular to L, this would mean very little; you can fairly easily convince yourself that every plane through P contains such a line.

Problem Set 8-1

1. The figure at the right represents plane E.
   a. Do any points outside the quadrilateral shown belong to plane E ?
   b. Is plane E intended to include every point outside the quadrilateral?



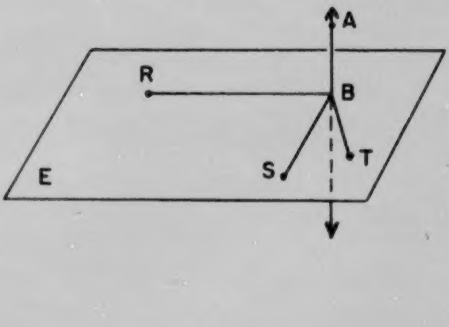
2. a. Sketch a plane perpendicular to a vertical line. (See Appendix V.)
   b. Sketch a plane perpendicular to a horizontal line.
   c. Does each of your sketches represent a line perpendicular to a plane?

3. a. Repeat the sketch of Problem 2b. Add to the sketch three lines in the plane which pass through the point of intersection. What is the relationship between each of the three lines and the original line?

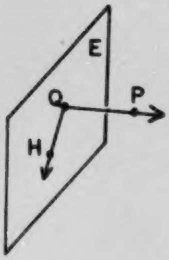
4. Reread the definition of perpendicularity between a line and a plane and decide whether the following statement is true if that definition is accepted:

   "If a line is perpendicular to a plane, then it is perpendicular to every line lying in the plane and passing through the point of intersection."

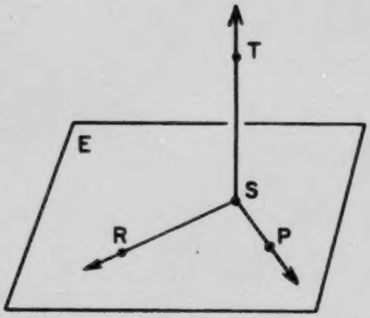
5. Given that B, R, S and T are in plane E, and that $\overleftrightarrow{AB} \perp E$, which of the following angles must be right angles:
   $\angle$ ABR, $\angle$ ABS, $\angle$ RBT, $\angle$ TBA, $\angle$ SBR?



6. If $\angle PQH$ is a right angle and Q and H are in E, should you infer from the definition of a line and a plane perpendicular that $\overleftrightarrow{PQ} \perp E$? Why or why not?

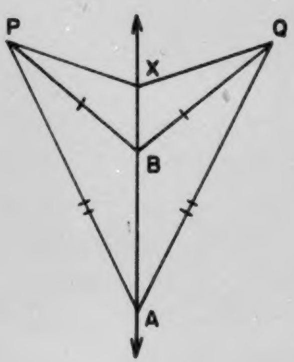


7. In the figure plane E contains points R, S, and P, but not T.
   a. Do points R, S and T determine a plane?
   b. If $\overrightarrow{SP}$ is perpendicular to the plane of R, S and T, which angles in the figure must be right angles?



8. a. If a point is equidistant from each of two other points, are the three points coplanar?
   b. If two points are each equidistant from each of two other points, are the four points coplanar?

*9. a. Given:
   Collinear points A, B and X as in the figure; B equidistant from P and Q; and A equidistant from P and Q.
   Prove: X is equidistant from P and Q.
   b. Does the proof require that Q be in the plane of A, B, X and P?

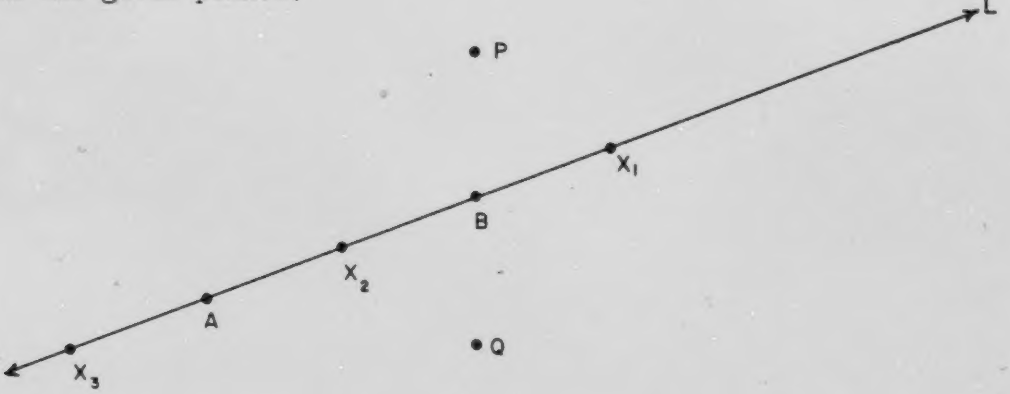


10. Look ahead to Theorem 8-1 and make a model for it from sticks, wire coat hangers, or straws.

8-2. The Basic Theorem.

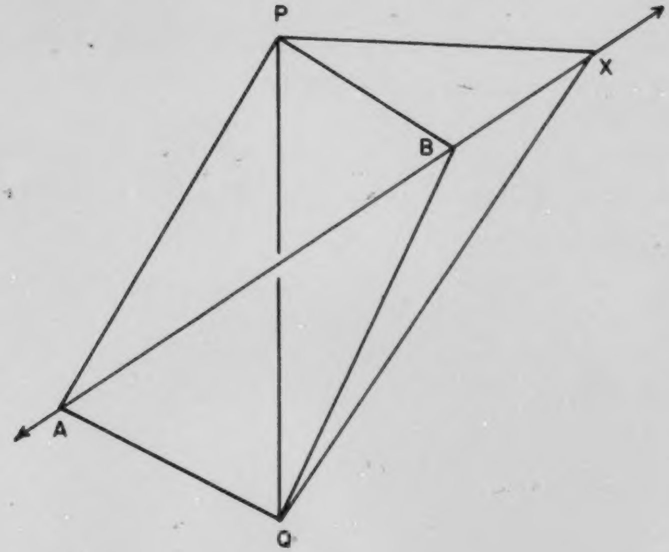
The basic theorem on perpendicularity in space says that if a plane E contains two lines, each perpendicular to a line L at the same point of L, then $L \perp E$. The proof of this is easier if we prove two preliminary theorems (lemmas).

Theorem 8-1. If each of two points of a line is equidistant from two given points, then every point of the line is equidistant from the given points.



Restatement: If P and Q are two points and L is a line such that two points A, B of L are each equidistant from P and Q, then every point X of L is equidistant from P and Q. (The above figure shows three possible positions for X. Of course, X might be at A or B.)

Proof: First we consider the case where X is on the same side of A as B. X might be at X_1, B, or X_2 but for convenience in the figure we show it beyond B at X_1. In this case $\angle PAB = \angle PAX$ and $\angle QAB = \angle QAX$. We treat this case in 3 steps.



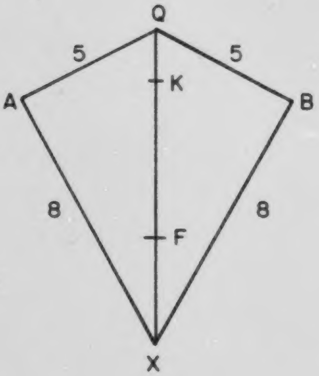
1. Since AP = AQ (given), BP = BQ (given), and AB = AB (identity), $\triangle ABP \cong \triangle ABQ$ (S.S.S.). Hence, $\angle PAB \cong \angle QAB$.

2. $\angle PAX \cong \angle QAX$. This is because $\angle PAB \cong \angle QAB$ by Step 1. (We are considering the case where $\angle PAX = \angle PAB$ and $\angle QAX = \angle QAB$.)

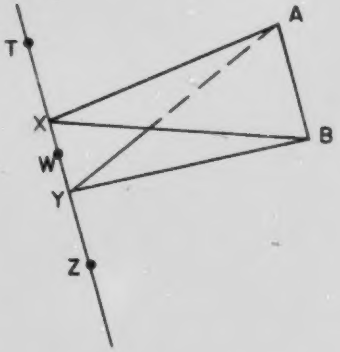
3. Using Step 2 and the facts that AP = AQ (given), and AX = AX (identity) we find that $\triangle PAX \cong \triangle QAX$ (S.A.S). Hence PX = QX.

The case where X lies on the ray opposite $\overrightarrow{AB}$ is proved in a similar fashion.

Problem Set 8-2a



1. A piece of paper AXBQ, as pictured here, is folded along $\overline{QX}$. Imagine A and B as both being in the foreground of the picture and $\overline{QX}$ in the background. Under these conditions will a point K of $\overline{QX}$ be equidistant from A and B? State a theorem to support your answer. If AF = 6, BF = _________.

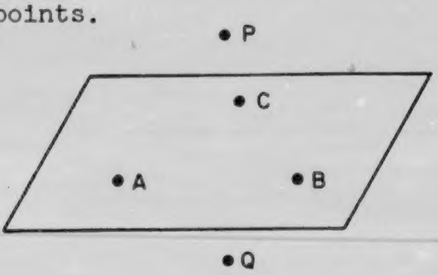


2. Here imagine plane AXB obscuring part of plane AYB. It is given that XA = XB and YA = YB. T, W and Z are three other points of $\overleftrightarrow{XY}$. Does TA = TB? Does WA = WB? Does ZA = ZB? State a theorem that supports each conclusion.

Theorem 8-2. If each of three non-collinear points of a plane is equidistant from two points, then every point of the plane is equidistant from these two points.

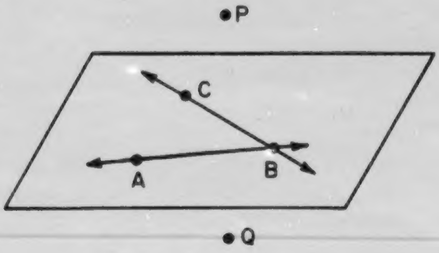
Given: Three non-collinear points A, B and C each equidistant from P and Q.

Prove: Every point of the plane determined by A, B and C is equidistant from P and Q.

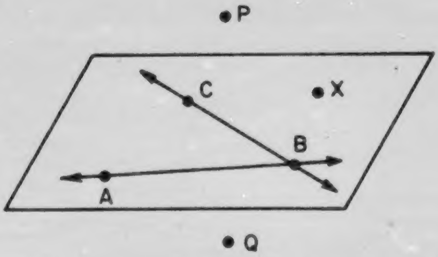


Proof: The proof is given in three steps.

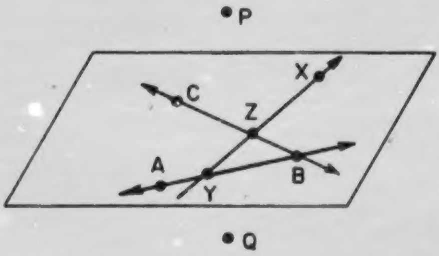
1. Since A and B are each given equidistant from P and Q, each point of $\overleftrightarrow{AB}$ is equidistant from P and Q. This follows from Theorem 8-1. Similarly each point of $\overleftrightarrow{BC}$ is equidistant from P and Q.



2. Let X be any other point of the plane. If X is on either $\overleftrightarrow{AB}$ or $\overleftrightarrow{CB}$, X is equidistant from P and Q by Step 1. If X is on one side of $\overleftrightarrow{BC}$, choose Y, some point of $\overleftrightarrow{AB}$ on the other side of $\overleftrightarrow{CB}$. The Plane Separation Postulate assures us that there is such a point Y and that $\overleftrightarrow{XY}$ will intersect $\overleftrightarrow{CB}$ in some point Z.



3. Since Z is on $\overleftrightarrow{CB}$ it is equidistant from P and Q by Step 1. Since Y is on $\overleftrightarrow{AB}$ it is equidistant from P and Q by Step 1. Therefore by Theorem 8-1 every point of YZ is equidistant from P and Q. X is one of these points.

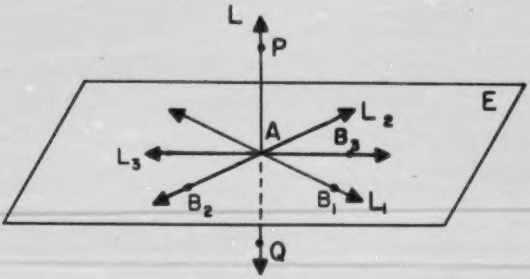


Since we have shown that each point X of the plane determined by A, B, C is equidistant from P and Q, Theorem 8-2 is established.

We are now ready to prove the basic theorem.

Theorem 8-3. If a line is perpendicular to each of two intersecting lines at their point of intersection, then it is perpendicular to the plane of these lines.

Restatement: Let L_1 and L_2 be lines in plane E intersecting at A and let L be a line through A perpendicular to L_1 and L_2. Then any line L_3 in E through A is perpendicular to L.

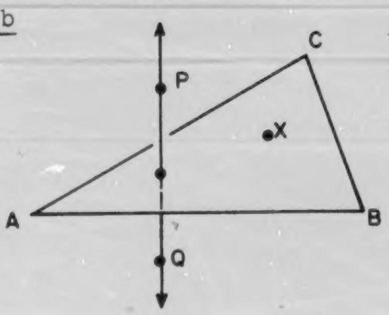


Proof:

Statement	Reason
1. Let P be a point on L, B_1 a point on L_1, B_2 a point on L_2, and B_3 a point on L_3, none of these points coinciding with A.	1. By the Ruler Postulate, each of these lines has an infinite number of points.
2. Let Q be the point on the ray opposite to $\overrightarrow{AP}$ such that AQ = AP.	2. Point Plotting Theorem.
3. In the plane containing L and L_1, L_1 is the perpendicular bisector of $\overline{PQ}$.	3. Definition of perpendicular bisector (Section 6-3).
4. B_1 is equidistant from P and Q.	4. Theorem 6-2.

5. B_2 is equidistant from P and Q.	5. Similar to 3 and 4.
6. A is equidistant from P and Q.	6. Step 2.
7. B_3 is equidistant from P and Q.	7. Steps 4, 5 and 6, and Theorem 8-2.
8. In the plane containing L and L_3, L_3 is the perpendicular bisector of $\overline{PQ}$.	8. Theorem 6-2.
9. $L \perp L_3$.	9. Definition of perpendicular bisector.
10. $L \perp E$.	10. Definition of perpendicularity of line and plane, since L_3 is any line in E through A.

Problem Set 8-2b



1. Suppose A, B and C are each equidistant from P and Q. Explain in terms of a definition or theorem why each point X of plane ABC is equidistant from P and Q.

2. Explain the relationship between the line of intersection L of two walls of your classroom and the plane of the floor. How many lines perpendicular to L could be drawn on the floor? Is L perpendicular to every line that could be drawn on the floor?

3.

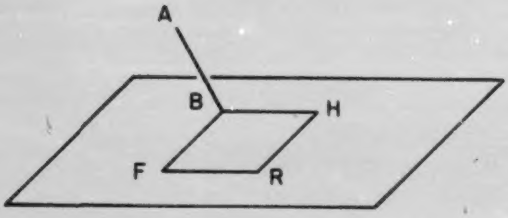
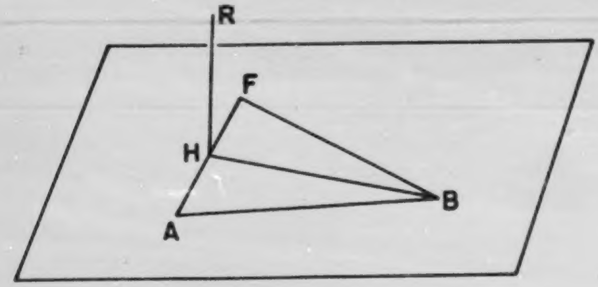


Figure FRHB is a square. $\overline{AB} \perp \overleftrightarrow{FB}$. A is not in plane FRHB.

a. How many planes are determined by pairs of segments in the figure? Name them.

b. At least one of the segments in this figure is perpendicular to one of the planes asked for in Part (a). Which segment? Which plane? A systematic approach to such a problem is to write down every pair of perpendicular segments you see in the figure. Then you can observe whether you have one line perpendicular to two intersecting lines.

4.

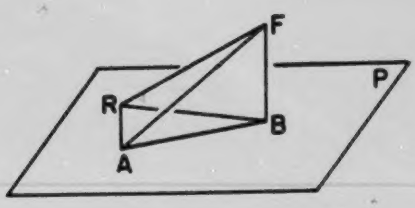


Δ ABF is isosceles with B as vertex. AH = FH. $\overline{RH} \perp \overline{HB}$. R is not in plane AFB.

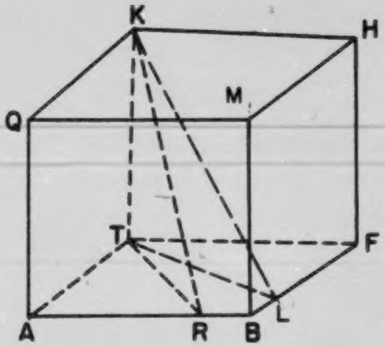
a. How many different planes are determined by the segments in the figure? Name them.

b. Do you find a segment that is perpendicular to a plane? If so, tell what segment and what plane and prove your statement.

5. In this figure, $\overline{FB} \perp$ plane P, and in Δ RAB, which lies in plane P, BR = BA. Prove Δ ABF $\cong \Delta$ RBF and $\angle$ FAR $\cong \angle$ FRA.



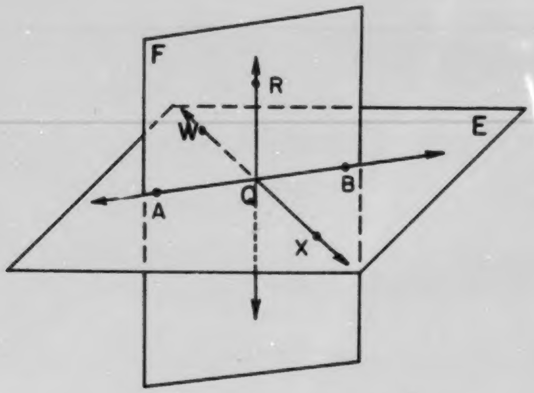
*6. Given the cube shown, with BR = BL. Does KR = KL? Prove that your answer is correct.



(Since we have not yet given a precise definition of a cube we state here, for use in your proof, the essential properties of the edges of a cube:

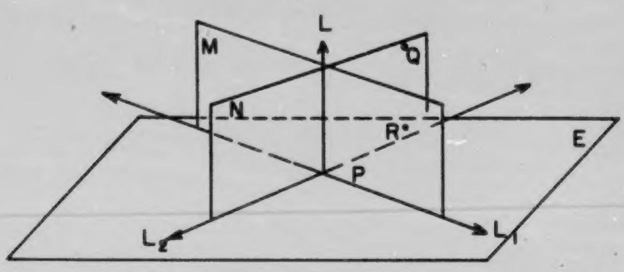
The edges of a cube consist of twelve congruent segments, related as shown in the picture, such that any two intersecting segments are perpendicular.)

7. In the accompanying figure $\overleftrightarrow{WX}$ is a line in plane E. Plane F $\perp$ $\overleftrightarrow{WX}$ at Q. In plane F, $\overleftrightarrow{RQ} \perp \overleftrightarrow{AB}$. $\overleftrightarrow{AB}$ is the intersection of E and F. Prove $\overleftrightarrow{RQ} \perp$ E.



For all we know up to now the conditions specified in the definition of a line and a plane perpendicular might be impossible to achieve. To reassure us, we need an existence theorem. The next theorem enables us to see that we are not talking about things that cannot exist in speaking of perpendicularity between lines and planes.

Theorem 8-4. Through a given point on a given line there passes a plane perpendicular to the line.



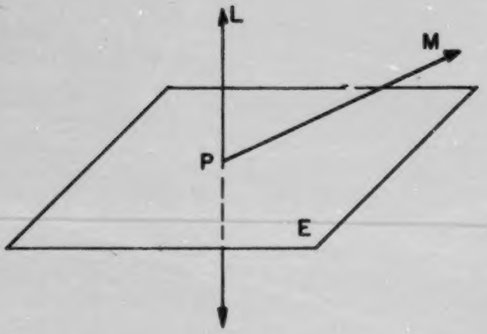
Proof: Let P be a point on a line L. We show in six steps that there is a plane E through P perpendicular to L.

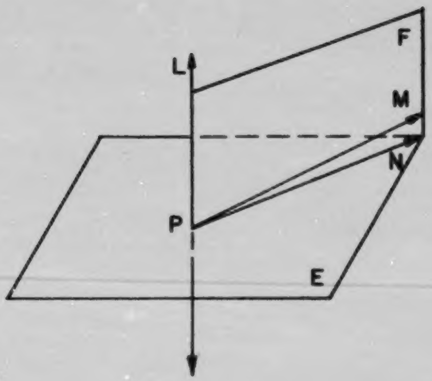
1. Let R be a point not on L. That there is such a point follows from Postulate 5a.
2. Let M be the plane determined by L and R. Theorem 3-3 tells us there is such a plane.
3. Let Q be a point not on M. Postulate 5b assures us that there is such a point.
4. Let N be the plane determined by L and Q.
5. In plane M there is a line L_1 perpendicular to L at P (Theorem 6-1), and in plane N there is a line L_2 perpendicular to L at P.
6. By Theorem 8-3, the plane E determined by L_1 and L_2 is perpendicular to L at P.

If $E \perp L$ at P then every line in E and through P is perpendicular to L, by definition. May there be some lines not in E but still perpendicular to L at P? The next theorem says, "No".

Theorem 8-5. If a line and a plane are perpendicular, then the plane contains every line perpendicular to the given line at its point of intersection with the given plane.

Restatement: If line L is perpendicular to plane E at point P, and if M is a line perpendicular to L at P, then M lies in E.





Proof:

Statements	Reasons
1. L and M determine a plane F.	1. Theorem 3-4.
2. Planes F and E intersect in a line N.	2. Postulate 8.
3. N $\perp$ L.	3. Definition of perpendicularity of line and plane.
4. M $\perp$ L.	4. Given.
5. M = N. (This means M and N are the same line.)	5. M and N both lie in plane F by Steps 1 and 2, are both $\perp$ L by Steps 3 and 4, but Theorem 6-1 says there is only one such perpendicular.
6. M lies in E.	6. M = N by Step 5 and N lies in E by Step 2.

This theorem enables us to prove the uniqueness theorem that goes with Theorem 8-4.

Theorem 8-6. Through a given point on a given line there is at most one plane perpendicular to the line..

Proof: Since a perpendicular plane contains all perpendicular lines through the point, and since two different planes have only one line in common (Theorem 3-4), there cannot be two such planes.

Just as in a plane where the characterization Theorem 6-2 followed the existence and uniqueness Theorem 6-1, so now we can prove a similar characterization theorem for space.

Theorem 8-7. The perpendicular bisecting plane of a segment is the set of all points equidistant from the end-points of the segment. Note that this theorem, like Theorem 6-2, has two parts.

Restatement: Let E be the perpendicular bisecting plane of $\overline{AB}$. Let C be the mid-point of $\overline{AB}$. Then

(1) If P is in E, then PA = PB, and

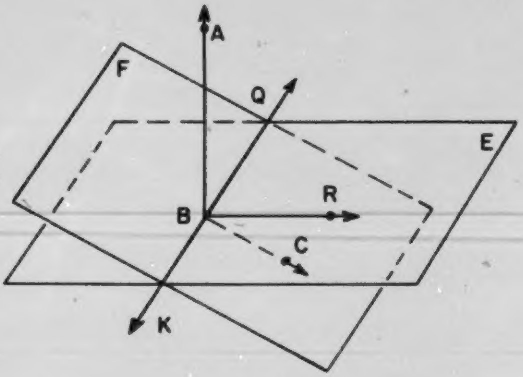
(2) If PA = PB, then P is in E.

The proof is left to the student.

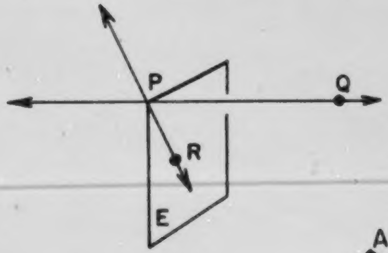
Problem Set 8-2c

1. a. At a point on a line how many lines are perpendicular to the line?
   b. At a point on a line how many planes are perpendicular to the line?

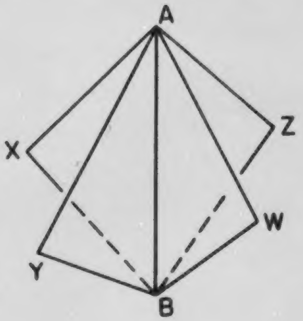
2. Planes E and F intersect in $\overleftrightarrow{KQ}$, as shown in this figure. $\overleftrightarrow{AB} \perp E$. $\overleftrightarrow{BR}$ lies in plane E. Plane ABR intersects F in $\overleftrightarrow{BC}$.
   Is $\overleftrightarrow{AB} \perp \overleftrightarrow{BR}$?
   Is $\overleftrightarrow{AB} \perp \overleftrightarrow{KQ}$.
   Is $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$?



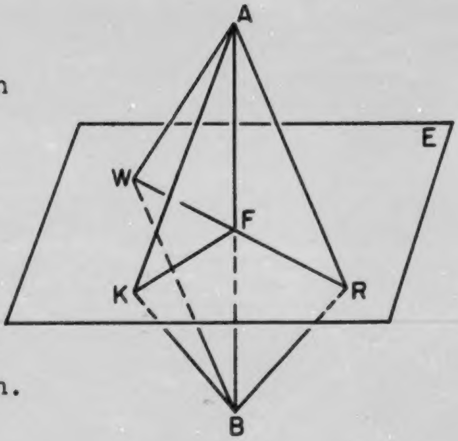
3. If $\overleftrightarrow{QP} \perp E$ at P and $\overleftrightarrow{QP} \perp \overleftrightarrow{PR}$, why does $\overleftrightarrow{PR}$ lie in E?



4. Assuming here that
   AX = BX,
   AY = BY,
   AW = BW,
   AZ = BZ,
   why are W, X, Y and Z coplanar?



5. Plane E is the perpendicular bisecting plane of $\overline{AB}$, as shown in the figure.



   a. $\overline{AW} \cong$ ________.
      $\overline{AK} \cong$ ________.
      $\overline{AR} \cong$ ________.
      $m\angle AFW =$ ________.
      $\angle AKF \cong$ ________.

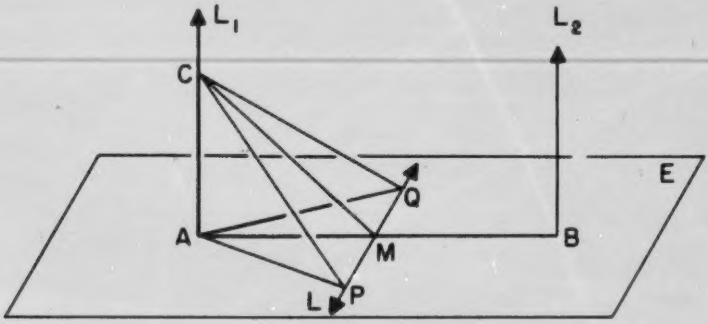
   b. Does FW = FK = FR? Explain.

*6. Prove this theorem: If L is a line intersecting plane E at point M, there is at least one line L' in E such that L' $\perp$ L.

The next theorem is a lemma which is useful in proving later theorems.

Theorem 8-8. Two lines perpendicular to the same plane are coplanar.

Proof: Let lines L_1 and L_2 be perpendicular to plane E at the points A and B respectively. Let M be the mid-point of $\overline{AB}$, let L be the line in E which is the perpendicular bisector of $\overline{AB}$, and let P and Q be two points on L such that PM = QM. Let C be a point on L_1 distinct from A.



1. By the S.A.S. Postulate, Δ AMP $\cong$ Δ AMQ, and so AP = AQ.
2. Since $L_1 \perp E$, $\angle$ CAP and $\angle$ CAQ are right angles, and the S.A.S. Postulate gives Δ CAP $\cong$ Δ CAQ, so that CP = CQ.
3. From AP = AQ and CP = CQ it follows, by Theorem 8-7, that C and A both lie in the bisecting plane E' of $\overline{PQ}$. Hence, L_1 lies in E'.
4. In exactly the same way we prove that L_2 lies in E'. Hence, L_1 and L_2 are coplanar.

8-3. Existence and Uniqueness Theorems.

The following theorems cover all possible relations between a point, a line and a perpendicular plane. They are stated here for completeness and for convenience in reference.

Theorem 8-9. Through a given point there passes one and only one plane perpendicular to a given line.

Theorem 8-10. Through a given point there passes one and only one line perpendicular to a given plane.

The proof of each of these theorems has two cases, depending on whether or not the given point lies on the given line or plane, and each case has two parts, one for proving existence and one for proving uniqueness. This makes a total of eight proofs required. Theorems 8-4 and 8-6 are two of these eight; the remaining six, some of which are hard and some easy, are given in Appendix VI.

Theorem 8-10 assures us of the existence of a unique perpendicular to a given plane from an external point. Hence, we are justified in giving the following definition, analogous to the one following Theorem 7-6.

Definition. The distance to a plane from an external point is the length of the perpendicular segment from the point to the plane.

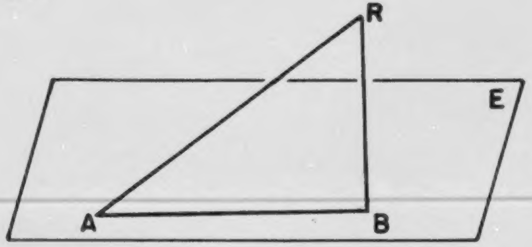
Theorem 8-11. The shortest segment to a plane from an external point is the perpendicular segment.

The proof is similar to that of Theorem 7-6.

Review Problems

1. Use a drawing if necessary to help you decide whether each statement is true or false.
   a. The intersection of two planes may be a segment.
   b. If a line intersects a plane in only one point, there are at least two lines in the plane perpendicular to the line.
   c. For any four points, there is a plane containing them all.
   d. If three lines intersect in pairs, but no point belongs to all three, the lines are coplanar.
   e. It is possible for three lines to intersect in a point, so that each is perpendicular to the other two.
   f. Only one line can be drawn perpendicular to a given line at a given point.
   g. At a point in a plane there is only one line perpendicular to the plane.
   h. The greatest number of regions into which three planes can separate space is eight.

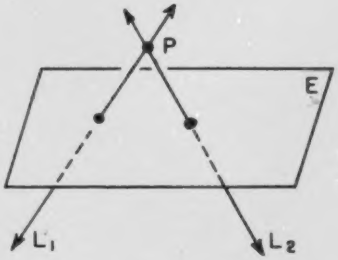
2. From a point R outside plane E, $\overline{RB} \perp E$ and $\overline{RB}$ intersects the plane in B. $\overline{RA}$ is any other segment from R, intersecting E in A. Compare the lengths $\overline{AR}$ and $\overline{RB}$. Compare the measures of $\angle A$ and $\angle B$.



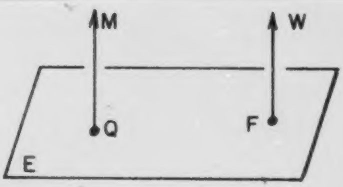
3. If the goal posts at one end of a football field are perpendicular to the ground, then they are coplanar even without a brace between them. Which theorem supports this conclusion? Can the goal posts still be coplanar even if they are not perpendicular to the ground? Could they fail to be coplanar even with a brace between them?

4. Do there always exist
   a. two lines perpendicular to a given line at a given point on the line?
   b. two planes perpendicular to a given line at a given point on the line?
   c. two lines perpendicular to a given plane at a given point on the plane?
   d. two planes perpendicular to a given line?
   e. two intersecting lines each perpendicular to a given plane?

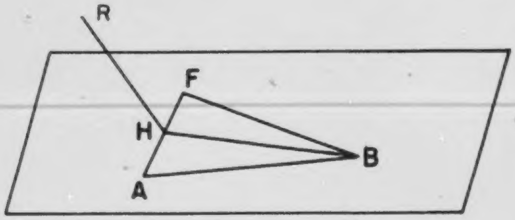
5. The assumption that two lines L_1 and L_2 are perpendicular to plane E and L_1 and L_2 intersect in point P not in plane E can be shown to be false by proving that the assumption leads to a contradiction of a theorem about figures in a plane. Which theorem?



6. Given $\overleftrightarrow{MQ} \perp$ plane E, and $\overleftrightarrow{WF} \perp$ to plane E. How many different planes are determined by $\overleftrightarrow{MQ}$, $\overleftrightarrow{MW}$, $\overleftrightarrow{WF}$ and $\overleftrightarrow{QF}$? Explain.



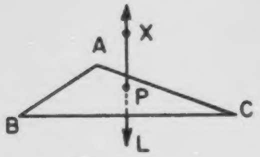
7.



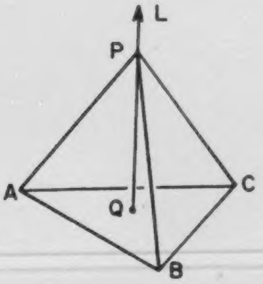
ΔABF is isosceles with vertex at B. HF = HA. $\overline{RH} \perp \overline{AF}$. R is not in the plane AFB.

   a. How many different planes are determined by the segments in the figure? Explain.
   b. Locate and describe a line that is perpendicular to a plane.

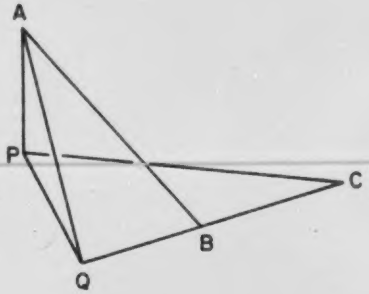
8. Given: P is in plane E which contains A, B, C; P is equidistant from A, B, C; line L $\perp$ E at P.
Prove: Every point, X, in L is equidistant from A, B, C.



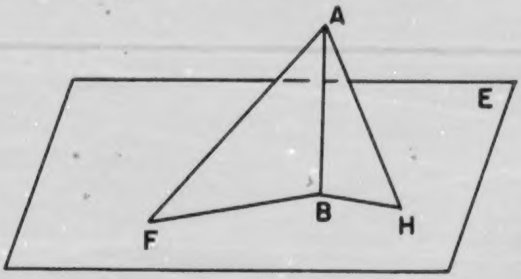
9. Given: Line L $\perp$ plane ABC at Q; point P of L is equidistant from A, B, C.
Prove: Every point of L is equidistant from A, B, C.
(Hint: Consider any point $X \neq Q$ on L and show $XA = XB = XC$.)



10. Given: $\overline{AP} \perp \overline{PQ}$ and $\overline{AP} \perp \overline{PC}$; $\overline{PQ} \perp \overleftrightarrow{BC}$ at Q.
Prove: $\overline{AQ} \perp \overleftrightarrow{BC}$.
(Hint: Take R on $\overleftrightarrow{QC}$ so that $QB = QR$. Draw $\overline{PB}$, $\overline{PR}$.)



11. Prove the following theorem: If from a point A outside a plane, a perpendicular $\overline{AB}$ and oblique (non-perpendicular) segments $\overline{AF}$ and $\overline{AH}$ are drawn, meeting the plane at unequal distances from B, the segment which meets the plane at the greater distance from B has the greater length.



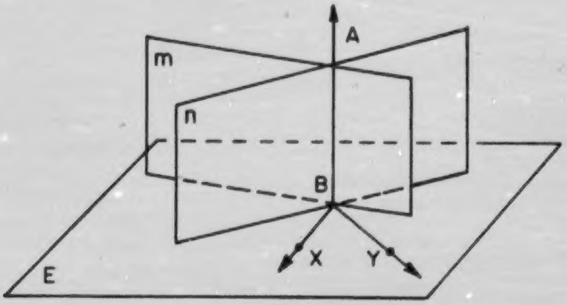
Given: $\overline{AB} \perp$ plane E. F and H are points of E such that $BF > BH$.

Prove: $AF > AH$.

12. Prove that each of four rays $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ and $\overrightarrow{AE}$ cannot be perpendicular to the other three.

13. Given: $\overleftrightarrow{XB}$ and $\overleftrightarrow{YB}$ are two lines in plane E; m is a plane $\perp \overleftrightarrow{XB}$ at B; n is a plane $\perp \overleftrightarrow{YB}$ at B; $\overleftrightarrow{AB}$ is the intersection of m and n.

Prove: $\overleftrightarrow{AB} \perp$ E.



Chapter 9

PARALLEL LINES IN A PLANE

9-1. Conditions Which Guarantee Parallelism.

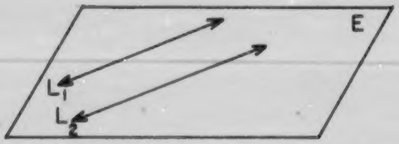
Thus far in our geometry we have been mainly concerned with what happens when lines and planes intersect in certain ways. We are now going to see what happens when they do not intersect. It will turn out that many more interesting things can be proved.

We first consider the case of two lines. Theorem 3-3 gives us some information right away, since it says that if two lines intersect they lie in a plane. Hence, if two lines are not coplanar they cannot intersect.

Definition: Two lines which are not coplanar are said to be skew.

You can easily find examples of skew lines in your classroom.

This still leaves open the question as to whether two coplanar lines must always intersect. In Theorem 9-2 we shall prove the existence of coplanar lines that do not intersect, but are parallel, like this:



Let us first make a precise definition.

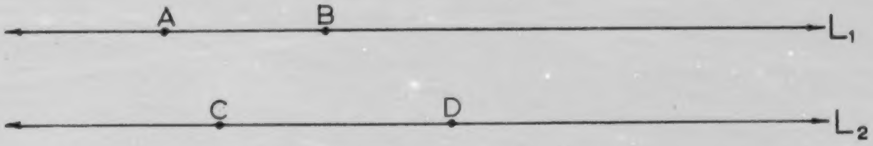
Definition: Two lines are parallel if they are coplanar and do not intersect.

Note that for two lines to be parallel two conditions must be satisfied: they must not intersect; they must both lie in the same plane.

Theorem 9-1. Two parallel lines lie in exactly one plane.

Proof: If L_1 and L_2 are parallel lines it follows from the above definition that there is a plane E containing L_1 and L_2. If P is any point of L_2 it follows from Theorem 3-3 that there is only one plane containing L_1 and P. Hence, E is the only plane containing L_1 and L_2.

We will use the abbreviation $L_1 \parallel L_2$ to mean that the lines L_1 and L_2 are parallel. As a matter of convenience we will say that two segments are parallel if the lines that contain them are parallel. We will speak similarly of a line and a segment, or a line and ray, and so on. For example, suppose we have given that $L_1 \parallel L_2$, in the figure below:



Then we can also write $\overline{AB} \parallel \overline{CD}$, $\overrightarrow{AB} \parallel L_2$, $L_1 \parallel \overrightarrow{CD}$, $\overrightarrow{BA} \parallel \overline{CD}$, and so on. Each of these statements is equivalent to the statement that $L_1 \parallel L_2$.

It does not seem easy to tell from the definition whether two lines which seem to be parallel really are parallel. Every line stretches out infinitely far in two directions, and to tell whether two lines do not intersect, we would have to look at all of each of the two lines. There is a simple condition, however, which is sufficient to guarantee that two lines are parallel. It goes like this:

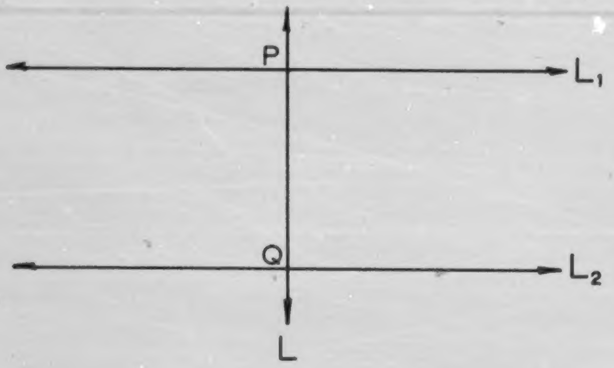
Theorem 9-2. Two lines in a plane are parallel if they are both perpendicular to the same line.

Proof: Suppose that L_1 and L_2 are two lines in plane E, each perpendicular to a line L, at points P and Q.

There are now two possibilities:

(1) L_1 and L_2 intersect in a point R.

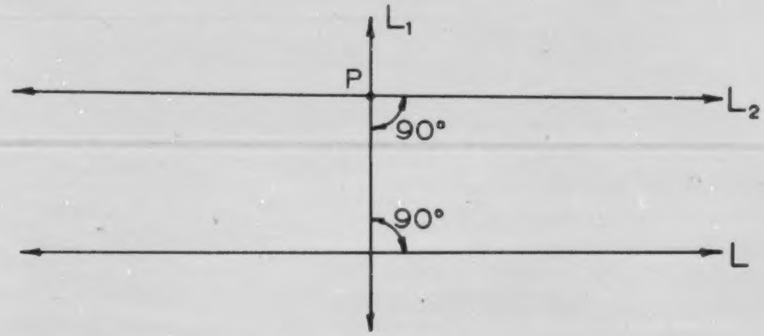
(2) L_1 and L_2 do not intersect.



In Case (1) we would have two lines, L_1 and L_2, each perpendicular to L and each passing through R. This is impossible by Theorem 6-1 if R lies on L, and by Theorem 6-3 if R is not on L. Hence, Case (2) is the only possible one, and so, by definition, $L_1 \parallel L_2$.

Theorem 9-2 enables us to prove the following important existence theorem.

Theorem 9-3. Let L be a line, and let P be a point not on L. Then there is at least one line through P, parallel to L.



Proof: Let L_1 be a line through P, perpendicular to L. (By Theorem 6-1, there is such a line.) Let L_2 be a line through P, perpendicular to L_1 in the plane of L and P. By Theorem 9-2, $L_2 \parallel L$.

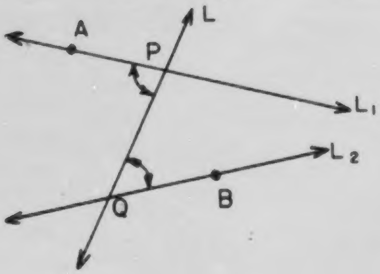
It might seem natural, at this point, to try to prove that the parallel given by Theorem 9-3 is unique; that is, we might try to show that in a plane through a given point not on a given line there is only one parallel to the given line. Astonishing as it may seem, this cannot be proved on the basis of the postulates that we have stated so far; it must be taken as a new postulate. We will discuss this in more detail in Section 9-3. In the meantime, before we get to work on the basis of this new postulate we shall prove some additional theorems which, like Theorem 9-2, tell us when two lines are parallel.

We first give some definitions.

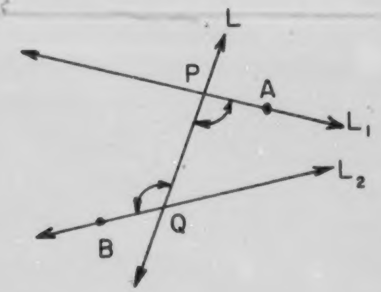
Definition: A transversal of two coplanar lines is a line which intersects them in two different points.

We say the two lines are "cut" by the transversal.

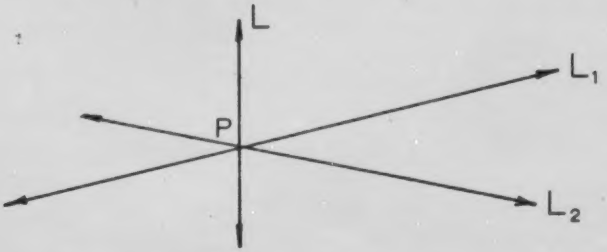
Definition: Let L be a transversal of L_1 and L_2, intersecting them in P and Q. Let A be a point of L_1 and B a point of L_2 such that A and B are on opposite sides of L. Then $\angle$ PQB and $\angle$ QPA are alternate interior angles formed by the transversal to the two lines.



OR



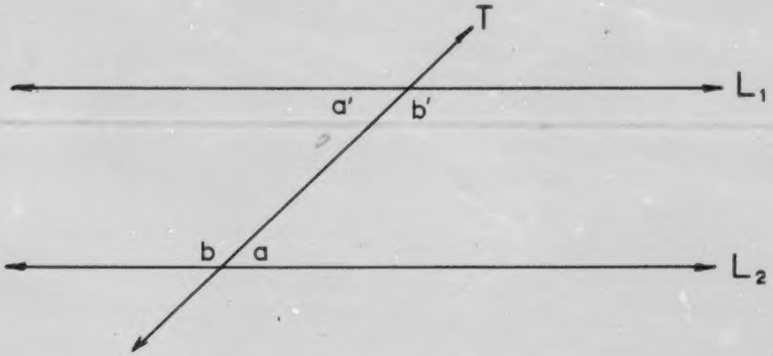
Notice that in the definition of a transversal, the two lines that we start with may or may not be parallel. But if they intersect, then the transversal is not allowed to intersect them at their common point. The situation in the figure below is not allowed:



That is, in this figure L is not a transversal to the lines L_1 and L_2.

Notice also that a common perpendicular to two lines in a plane, as in Theorem 9-2, is always a transversal.

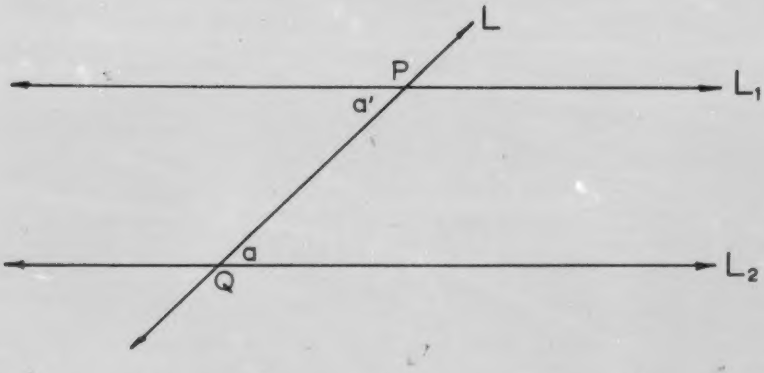
Theorem 9-4. If two lines are cut by a transversal, and if one pair of alternate interior angles are congruent, then the other pair of alternate interior angles are also congruent.



That is, if $\angle a \cong \angle a'$, then $\angle b \cong \angle b'$. And if $\angle b \cong \angle b'$, then $\angle a \cong \angle a'$. The proof is left to the student.

The following theorem is a generalization of Theorem 9-2, that is, it includes Theorem 9-2 as a special case:

Theorem 9-5. If two lines are cut by a transversal, and if a pair of alternate interior angles are congruent, then the lines are parallel.

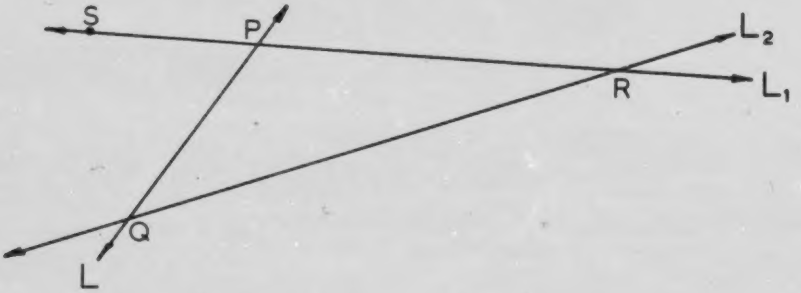


Proof: Let L be a transversal to L_1 and L_2, intersecting them in P and Q. Suppose that a pair of alternate interior angles are congruent. There are now two possibilities:

(1) L_1 and L_2 intersect in a point R.

(2) $L_1 \parallel L_2$.

In Case (1) the figure looks like this:



Let S be a point of L_1 on the opposite side of L from R. Then $\angle$ SPQ is an exterior angle of $\triangle$ PQR, and $\angle$ PQR is one of the remote interior angles. By Theorem 7-1, this means that

$$m \angle SPQ > m \angle PQR.$$

But we know by hypothesis that one pair of alternate interior angles are congruent. By the preceding theorem, both pairs of alternate interior angles are congruent. Therefore,

$$m \angle SPQ = m \angle PQR.$$

Since Statement (1) leads to a contradiction of our hypothesis, Statement (1) is false. Therefore Statement (2) is true.

Problem Set 9-1

1. a. Does the definition of parallel lines state that the lines must remain the same distance apart?

   b. If two given lines do not lie in one plane, can the lines be parallel?

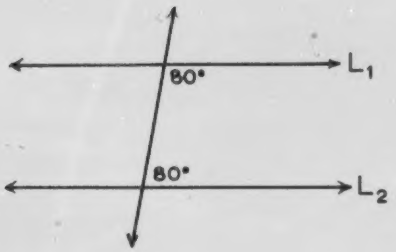
2. Two lines in a plane are parallel if ______, or if ______, or if ________.

3. If two lines in a plane are intersected by a transversal, are the alternate interior angles always congruent?

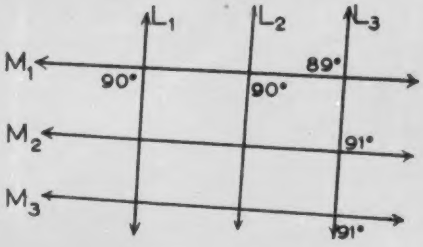
4. In space, if two lines are perpendicular to a third line, are the two lines parallel?

5. a. If the 80° angles were correctly drawn, would L_1 be parallel to L_2 according to Theorem 9-5? Explain.

   b. How many different measures of angles would occur in the drawing? What measures?



6. In the figure, if the angles were of the size indicated, which lines would be parallel?



7. Given a line L and a point P not on L, show how protractor and ruler can be used to draw a parallel to L through P.

8. Suppose the following two definitions are agreed upon:

A vertical line is one containing the center of the earth.

A horizontal line is one which is perpendicular to some vertical line.

a. Could two horizontal lines be parallel?

b. Could two vertical lines be parallel?

c. Could two horizontal lines be perpendicular?

d. Could two vertical lines be perpendicular?

e. Would every vertical line also be horizontal?

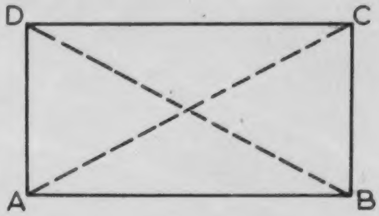
f. Would every horizontal line also be vertical?

g. Could a horizontal line be parallel to a vertical line?

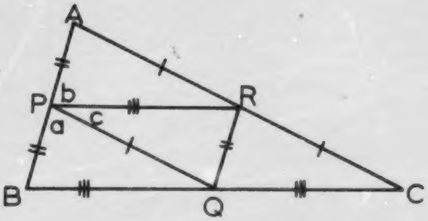
h. Would every line be horizontal?

9. Is it possible to find two lines in space which are neither parallel nor intersecting?

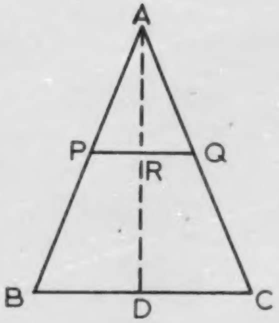
10. Given: $m\angle DAB = m\angle CBA = 90$,
and $AD = CB$.
Prove: $m\angle ADC = m\angle BCD$.
Can you also prove $m\angle ADC = m\angle BCD = 90$?



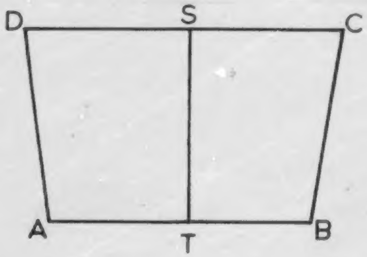
11. Given the figure with
AR = RC = PQ,
AP = PB = RQ,
BQ = QC = PR.
Prove:
m $\angle$ A + m $\angle$ B + m $\angle$ C = 180.
(Hint: Prove m $\angle$ a = m $\angle$ A,
m $\angle$ b = m $\angle$ B, m $\angle$ c = m $\angle$ C.)



12. Given: AB = AC, AP = AQ.
Prove: $\overleftrightarrow{PQ} \parallel \overleftrightarrow{BC}$.
(Hint: Let the bisector of
$\angle$ A intersect $\overline{PQ}$ at R
and $\overline{BC}$ at D.)

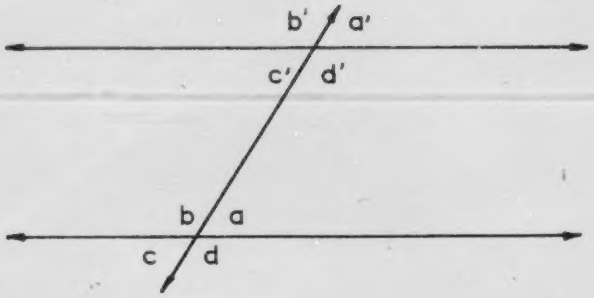


13. Given: The figure with
$\angle$ A $\cong$ $\angle$ B,
AD = BC,
AT = TB,
SD = SC.
Prove:
$\overline{ST} \perp \overline{DC}$.
$\overline{ST} \perp \overline{AB}$.
$\overline{DC} \parallel \overline{AB}$.



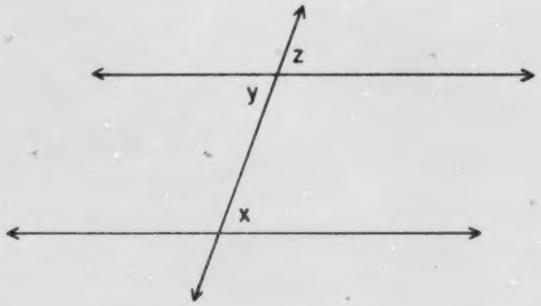
9-2. Corresponding Angles.

In the figure below, the angles marked a and a' are called corresponding angles:



Similarly, b and b' are corresponding angles; and the pairs c, c' and d, d' are also corresponding angles.

Definition: If two lines are cut by a transversal, if $\angle x$ and $\angle y$ are alternate interior angles, and if $\angle y$ and $\angle z$ are vertical angles, then $\angle x$ and $\angle z$ are corresponding angles.



You should prove the following theorem.

Theorem 9-6. If two lines are cut by a transversal, and if one pair of corresponding angles are congruent, then the other three pairs of corresponding angles have the same property.

The proof is only a little longer than that of Theorem 9-4.

Theorem 9-7. If two lines are cut by a transversal, and if a pair of corresponding angles are congruent, then the lines are parallel. The proof is left to the student.

It looks as though the converses of Theorem 9-5 and Theorem 9-7 ought to be true. The converse of Theorem 9-5 would say that if two parallel lines are cut by a transversal, then the alternate interior angles are congruent. The converse of Theorem 9-7 would say that if two parallel lines are cut by a transversal, then corresponding angles are congruent. These theorems, however, cannot be proved on the basis of the postulates that we have stated so far. To prove them, we shall need to use the Parallel Postulate, which will be stated in the next section.

The Parallel Postulate is essential to the proofs of many other theorems of our geometry as well. Some of these you are already familiar with from your work in other grades. For example, you have known for some time that the sum of the measures of the angles of any triangle is 180. Yet, without the Parallel Postulate it is impossible to prove this very important theorem. Let us go on, then, to the Parallel Postulate.

9-3. The Parallel Postulate.

Postulate 16. (The Parallel Postulate.) Through a given external point there is at most one line parallel to a given line.

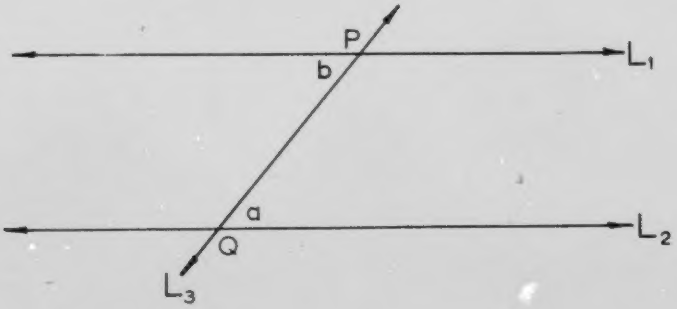
Notice that we don't need to say, in the postulate, that there is at least one such parallel, because we already know this by Theorem 9-3.

It might seem natural to suppose that we already have enough postulates to be able to prove anything that is "reasonable"; and since the Parallel Postulate is reasonable, we might try to prove it instead of calling it a postulate. At any rate, some very clever people felt this way about the postulate, over a period of a good many centuries. None of them, however, was able to find a proof. Finally, in the last century, it was discovered that no such proof is possible. The point is that there are some mathematical systems that are almost like the geometry that we are studying, but not quite. In these mathematical systems, nearly all of the postulates of ordinary geometry are satisfied, but the Parallel Postulate is not. These "Non-Euclidean Geometries" may seem strange, and in fact they are. (For example, in these "geometries" there is no such thing as a square.) Not only do they lead to interesting mathematical theories, but they also have important applications to physics.

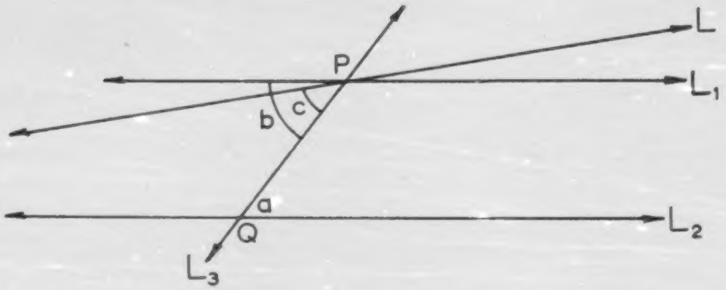
Now that we have the Parallel Postulate we can go on to prove numerous important theorems we could not prove without it. We start by proving the converse of Theorem 9-5.

Theorem 9-8. If two parallel lines are cut by a transversal, then alternate interior angles are congruent.

Proof: We have given parallel lines L_1 and L_2, and a transversal L_3, intersecting them in P and Q.



Suppose that $\angle a$ and $\angle b$ are not congruent. Let L be a line through P for which alternate interior angles are congruent.



(By the Angle Construction Postulate, there is such a line.)

Then $L \neq L_1$, because $\angle b$ and $\angle c$ are not congruent.

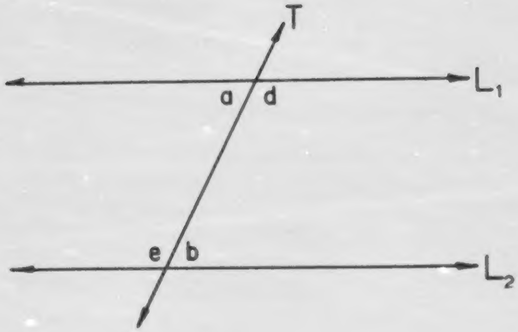
Now let us see what we have. By hypothesis, $L_1 \parallel L_2$. And by Theorem 9-5, we know that $L \parallel L_2$. Therefore there are two lines through P, parallel to L_2. This is impossible, because it contradicts the Parallel Postulate. Therefore $\angle a \cong \angle b$, which was to be proved.

The proofs of the following theorems are short, and you should write them for yourself:

Theorem 9-9. If two parallel lines are cut by a transversal, each pair of corresponding angles are congruent.

Theorem 9-10. If two parallel lines are cut by a transversal, interior angles on the same side of the transversal are supplementary.

Restatement: Given $L_1 \parallel L_2$ and T intersects L_1 and L_2. Prove that $\angle b$ is supplementary to $\angle d$ and $\angle a$ is supplementary to $\angle e$.

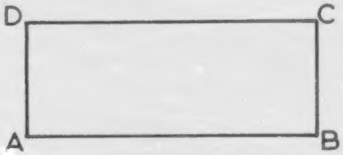


Theorem 9-11. In a plane, two lines parallel to the same line are parallel to each other.

Theorem 9-12. In a plane, if a line is perpendicular to one of two parallel lines it is perpendicular to the other.

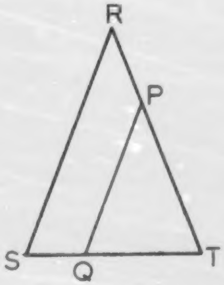
Problem Set 9-3

1. Given:
   $m \angle A = m \angle B = m \angle C = 90.$
   Prove: $m \angle D = 90.$



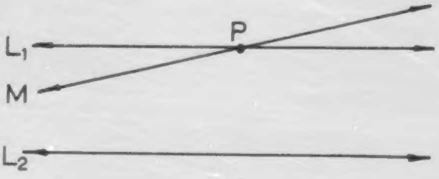
2. Prove that a line parallel to the base of an isosceles triangle and intersecting the other two sides of the triangle forms another isosceles triangle.

3. Given: In the figure,
   $RT = RS$, $\overleftrightarrow{PQ} \parallel \overleftrightarrow{RS}$.
   Prove: $PQ = PT$.

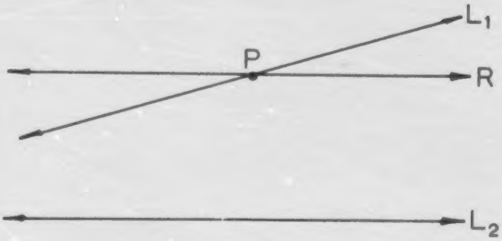


4. Review indirect proof as illustrated by the proof of Theorem 9-8. Give an indirect proof of each of the following statements, showing a contradiction of the Parallel Postulate.

a. In a plane, if a third line M intersects one of two parallel lines L_1 at P, it also intersects the other L_2.



b. In a plane, if a line R intersects only one of two other lines L_1 and L_2, then the lines L_1 and L_2 intersect.



Given: R intersects L_1 at P.

R does not intersect L_2.

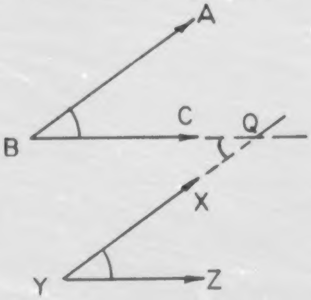
Prove: L_1 intersects L_2

5. a. Prove: Two angles in a plane which have their sides respectively parallel and extending both in the same (or both in opposite) directions are congruent.

Given: $BA \parallel \overrightarrow{YX}$,

$\overrightarrow{BC} \parallel \overrightarrow{YZ}$.

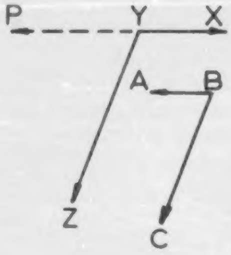
Prove: $\angle ABC \cong \angle XYZ$



b. Prove: Two angles in a plane which have their sides respectively parallel but have only one pair extending in the same direction are supplementary.

Given: $\overrightarrow{BA} \parallel \overrightarrow{YX}$,
$\overrightarrow{BC} \parallel \overrightarrow{YZ}$.

Prove: In (a) $\angle ABC \cong \angle XYZ$.
In (b) $m\angle ABC + m\angle XYZ = 180$.



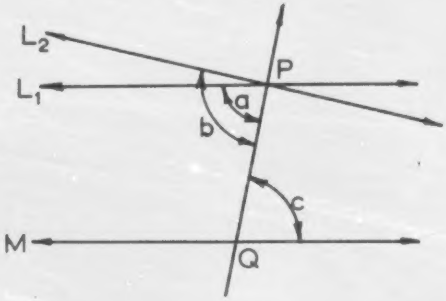
(Note: Only certain cases are illustrated and proved here. All other cases can also be proved easily. The term "direction" is undefined but should be understood.)

6. Make drawings of various pairs of angles ABC and DEF such that $\overrightarrow{BA} \perp \overrightarrow{ED}$ and $\overrightarrow{BC} \perp \overrightarrow{EF}$. State a theorem that you think may be true about the measures of such angles.

*7. If Theorem 9-8 is assumed as a postulate, then the Parallel Postulate can be proved as a theorem. (That is, it must be shown that there cannot be a second parallel to a line through a point not on it.)

Given: L_1 and L_2 are two lines containing P, and $L_1 \parallel M$.

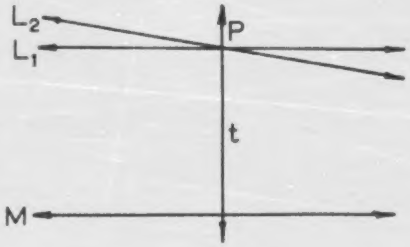
Prove: L_2 not parallel to M.



*8. Show that if Theorem 9-12 (If a transversal is perpendicular to one of two parallel lines, it is perpendicular to the other.) is assumed as a postulate, the Parallel Postulate can be proved as a theorem.

Given: $L_1 \parallel M$ and L_1 and L_2 contain P. ($L_2 \neq L_1$.)

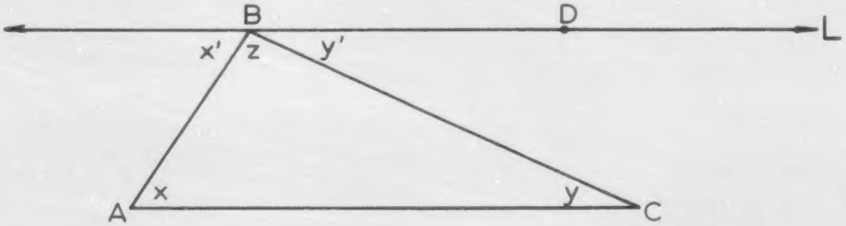
Prove: L_2 not parallel to M.



9-4. Triangles.

Theorem 9-13. The sum of the measures of the angles of a triangle is 180.

Proof: Given $\triangle ABC$, let L be the line through B, parallel to $\overline{AC}$. Let $\angle x$, $\angle x'$, $\angle y$, $\angle y'$ and $\angle z$ be as in the figure.



Let D be a point of L on the same side of $\overleftrightarrow{AB}$ as C. Since $\overline{AC} \parallel \overleftrightarrow{BD}$, A is on the same side of $\overleftrightarrow{BD}$ as C. Therefore C is in the interior of $\angle ABD$ (definition of interior of an angle), and so, by the Angle Addition Postulate, we have

$$m\angle ABD = m\angle z + m\angle y'.$$

By the Supplement Postulate,

$$m\angle x' + m\angle ABD = 180.$$

Therefore

$$m\angle x' + m\angle z + m\angle y' = 180.$$

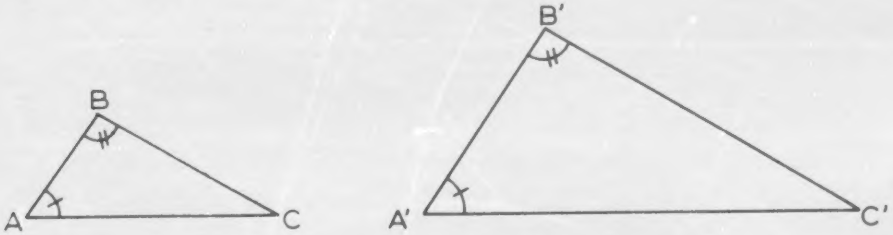
But we know by Theorem 9-8 that $m\angle x = m\angle x'$ and $m\angle y = m\angle y'$, because these are alternate interior angles. By substitution we get

$$m\angle x + m\angle z + m\angle y = 180,$$

which was to be proved.

From this we get a number of important corollaries:

Corollary 9-13-1. Given a correspondence between two triangles. If two pairs of corresponding angles are congruent, then the third pair of corresponding angles are also congruent.



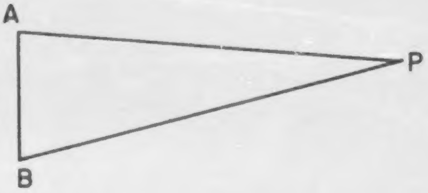
The corollary says that if $\angle A \cong \angle A'$ and $\angle B \cong \angle B'$, then $\angle C \cong \angle C'$. As the figure suggests, the corollary applies to cases where the correspondence given is not a congruence, as well as to cases where $\triangle ABC \cong \triangle A'B'C'$.

Corollary 9-13-2. The acute angles of a right triangle are complementary.

Corollary 9-13-3. For any triangle, the measure of an exterior angle is the sum of the measures of the two remote interior angles.

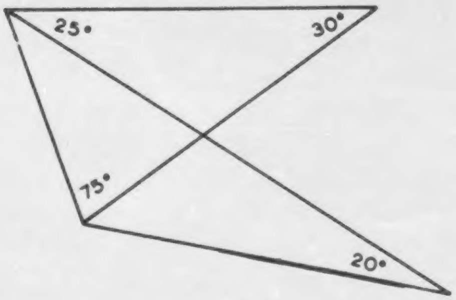
Problem Set 9-4

1. If the measures of two angles of a triangle are as follows, what is the measure of the third angle?
    - a. 37 and 58.
    - b. 149 and 30.
    - c. n and n.
    - d. r and s.
    - e. $45 + a$ and $45 - a$.
    - f. 90 and $\frac{1}{2}k$.
2. To find the distance from a point A to a distant point P, a surveyor may measure a small distance AB and also measure $\angle A$ and $\angle B$. From this information he can compute the measure of $\angle P$ and by appropriate formulas then compute AP. If $m\angle A = 87.5$ and $m\angle B = 88.3$, compute $m\angle P$.



3. Why is the Parallel Postulate essential to the proof of Theorem 9-13?

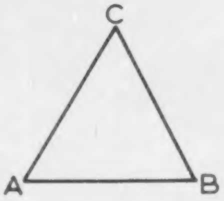
4. On a drawing like the one on the right fill in the values of all of the angles.

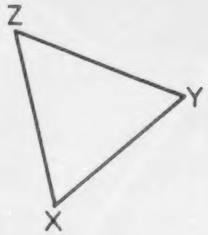


5. Given: $\angle A \cong \angle X$ and $\angle B \cong \angle Y$, can you correctly conclude that:

   a. $\angle C \cong \angle Z$?

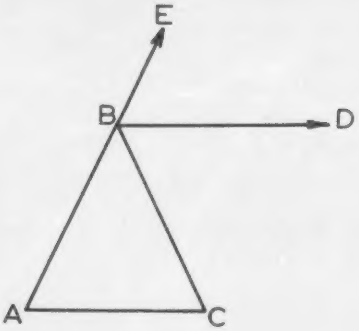
   b. $\overline{AB} \cong \overline{XY}$?





6. Given: $\overrightarrow{BD}$ bisects $\angle EBC$, and $\overleftrightarrow{BD} \parallel \overleftrightarrow{AC}$.

   Prove: $AB = BC$.

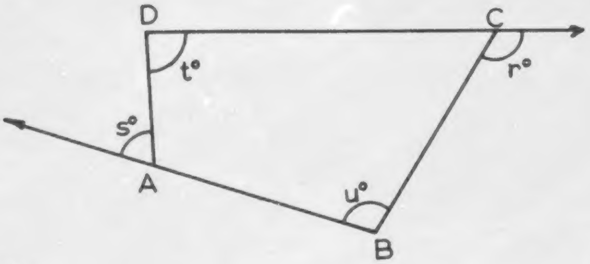


7. The bisector of an exterior angle at the vertex of an isosceles triangle is parallel to the base. Prove this.

8. Given: The figure.

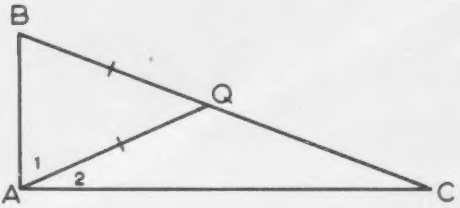
Prove: $s + r = t + u$.

(Hint: Draw $\overline{DB}$.)



*9. Given: In the figure, $\angle BAC$ is a right angle and $QB = QA$.

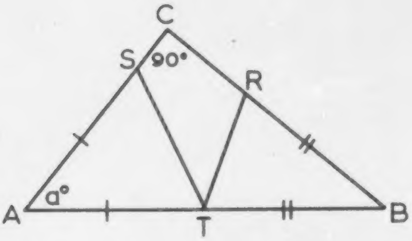
Prove: $QB = QC$.



*10. Given: In $\triangle ABC$, $\angle C$ is a right angle, $AS = AT$ and $BR = BT$.

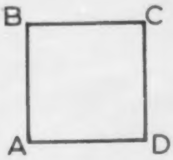
Prove: $m\angle STR = 45$.

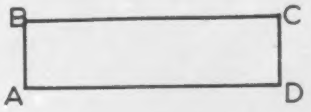
(Hint: Suppose $m\angle A = a$. Write formulas in turn for the measures of other angles in the figure in terms of a.)

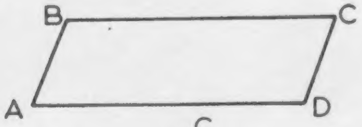


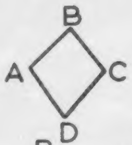
9-5. Quadrilaterals in Plane.

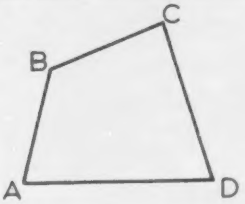
A quadrilateral is a plane figure with four sides, like one of the following:

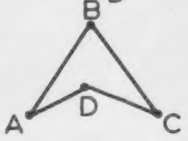










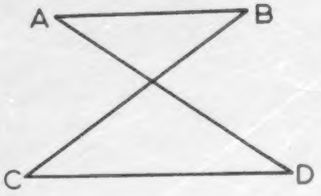


The two figures on the bottom illustrate what we might call the most general case, in which no two sides are congruent, no two sides are parallel, and no two angles are congruent.

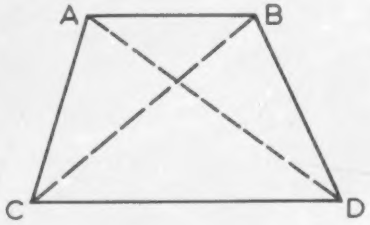
We can state the definition of a quadrilateral more precisely, in the following way.

Definition: Let A, B, C and D be four points lying in the same plane, such that no three of them are collinear, and such that the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$ intersect only in their end-points. Then the union of these four segments is a quadrilateral.

For short, we will denote this figure by ABCD. Notice that in each of the examples above, with the exception of the last one, the quadrilateral plus its interior forms a convex set, in the sense which was defined in Chapter 3. This is not true of the figure at the lower right, but this figure is still a quadrilateral under our definition. Notice, however, that under our definition of a quadrilateral, figures like the following one are ruled out.

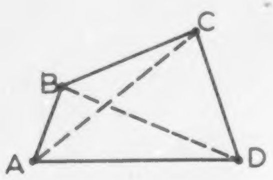


Here the figure is not a quadrilateral, because the segments $\overline{BC}$ and $\overline{DA}$ intersect in a point which is not an end-point of either of them. Notice also, however, that a quadrilateral can be formed, using these same four points as vertices, like this:



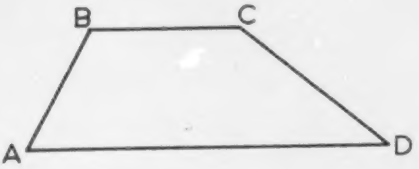
Here ABDC is a quadrilateral.

Definitions: *Opposite sides* of a quadrilateral are two sides that do not intersect. Two of its *angles* are *opposite* if they do not contain a common side. Two sides are called *consecutive* if they have a common vertex. Similarly, two angles are called *consecutive* if they contain a common side. A *diagonal* is a segment joining two non-consecutive vertices.

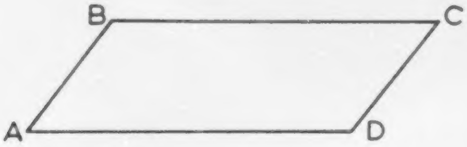


In a quadrilateral ABCD, $\overline{AB}$ and $\overline{CD}$ are opposite sides, as are $\overline{BC}$ and $\overline{AD}$. $\overline{AD}$ and $\overline{CD}$ or $\overline{AD}$ and $\overline{AB}$ are consecutive sides. $\overline{AC}$ and $\overline{BD}$ are the diagonals of ABCD. Which angles are opposite? Which consecutive?

Definition: A trapezoid is a quadrilateral in which two, and only two, opposite sides are parallel.



Definition: A parallelogram is a quadrilateral in which both pairs of opposite sides are parallel.



You should not have much trouble in proving the basic theorems on trapezoids and parallelograms:

Theorem 9-14. Either diagonal separates a parallelogram into two congruent triangles. That is, if ABCD is a parallelogram, then $\triangle ABC \cong \triangle CDA$.

Theorem 9-15. In a parallelogram, any two opposite sides are congruent.

Corollary 9-15-1. If $L_1 \parallel L_2$ and if P and Q are any two points on L_1, then the distances of P and Q from L_2 are equal.

This property of parallel lines is sometimes abbreviated by saying that "parallel lines are everywhere equidistant".

Definition: The distance between two parallel lines is the distance from any point of one line to the other line.

Theorem 9-16. In a parallelogram, any two opposite angles are congruent.

Theorem 9-17. In a parallelogram, any two consecutive angles are supplementary.

Theorem 9-18. The diagonals of a parallelogram bisect each other.

In Theorems 9-14 through 9-18 we are concerned with several properties of a parallelogram; that is, if we know that a quadrilateral is a parallelogram we can conclude certain facts about it. In the following three theorems we provide for the converse relationship; that is, if we know certain facts about a quadrilateral we can conclude that it is a parallelogram.

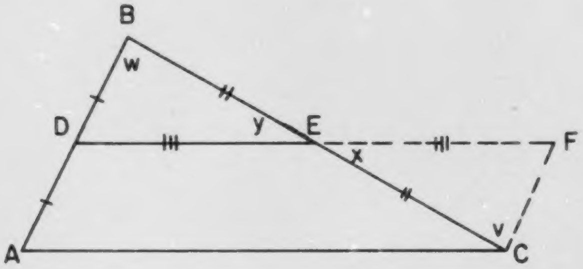
Theorem 9-19. Given a quadrilateral in which both pairs of opposite sides are congruent. Then the quadrilateral is a parallelogram.

Theorem 9-20. If two sides of a quadrilateral are parallel and congruent, then the quadrilateral is a parallelogram.

Theorem 9-21. If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

The following theorem states two useful facts. The proof of this theorem is given in full.

Theorem 9-22. The segment between the mid-points of two sides of a triangle is parallel to the third side and half as long as the third side.



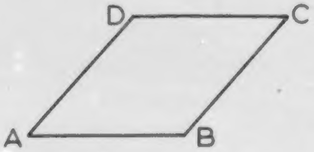
Restatement: Given $\triangle ABC$. Let D and E be the mid-points of $\overline{AB}$ and $\overline{BC}$. Then $DE \parallel AC$, and $DE = \frac{1}{2} AC$.

Proof: Using the Point Plotting Theorem, let F be the point of the ray opposite to $\overrightarrow{ED}$ such that $EF = DE$. We give the rest of the proof in the two-column form. The notation for angles is that of the figure.

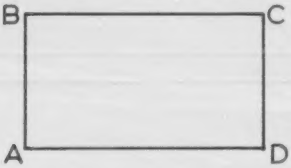
Statements	Reasons
1. $EF = ED$.	1. F was chosen so as to make this true.
2. $EB = EC$.	2. E is the mid-point of $\overline{BC}$.
3. $\angle x \cong \angle y$.	3. Vertical angles are congruent.
4. $\triangle EFC \cong \triangle EDB$.	4. The S.A.S. Postulate.
5. $\angle v \cong \angle w$.	5. Corresponding parts of congruent triangles.
6. $\overleftrightarrow{AB} \parallel \overleftrightarrow{CF}$.	6. Theorem 9-5.
7. $AD = FC$.	7. $AD = DB$, by hypothesis, and $DB = FC$, by statement 4.
8. ADFC is a parallelogram.	8. Theorem 9-20.
9. $\overline{DE} \parallel \overline{AC}$.	9. Definition of a parallelogram.
10. $DE = \frac{1}{2} AC$.	10. $DE = \frac{1}{2} DF$, by statement 1, and $DF = AC$, by Theorem 9-15.

9-6. Rhombus, Rectangle and Square.

Definitions: A rhombus is a parallelogram all of whose sides are congruent.



A rectangle is a parallelogram all of whose angles are right angles.



Finally, a square is a rectangle all of whose sides are congruent.

As before, we leave the proofs of the following theorems for the student.

Theorem 9-23. If a parallelogram has one right angle, then it has four right angles, and the parallelogram is a rectangle.

Theorem 9-24. In a rhombus, the diagonals are perpendicular to one another.

Theorem 9-25. If the diagonals of a quadrilateral bisect each other and are perpendicular, then the quadrilateral is a rhombus.

Problem Set 9-6

1. For which of the quadrilaterals -- rectangle, square, rhombus, parallelogram -- can each of the following properties be proved?

   a. Both pairs of opposite angles are congruent.

   b. Both pairs of opposite sides are congruent.

   c. Each diagonal bisects two angles.

   d. The diagonals bisect each other.

   e. The diagonals are perpendicular.

   f. Each pair of consecutive angles is supplementary.

   g. Each pair of consecutive sides is congruent.

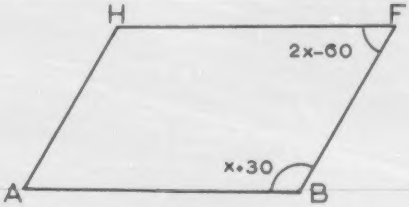
   h. The figure is a parallelogram.

   i. Each pair of consecutive angles is congruent.

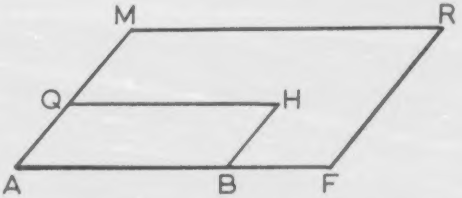
   j. The diagonals are congruent.

2. With the measures of the angles as given in parallelogram ABFH, give the degree measure of each angle.

   $m\angle A =$ ____.
   $m\angle B =$ ____.
   $m\angle F =$ ____.
   $m\angle H =$ ____.



3. In this figure ABHQ and AFRM are parallelograms. What is the relationship of $\angle M$ to $\angle H$? of $\angle R$ to $\angle H$? Prove your answer.



4. Would the following information about a quadrilateral be sufficient to prove it a parallelogram? a rectangle? a rhombus? a square? Consider each item of information separately.

   a. Both pairs of its opposite sides are parallel.

   b. Both pairs of its opposite sides are congruent.

   c. Three of its angles are right angles.

   d. Its diagonals bisect each other.

   e. Its diagonals are congruent.

   f. Its diagonals are perpendicular and congruent.

   g. Its diagonals are perpendicular bisectors of each other.

   h. It is equilateral.

   i. It is equiangular.

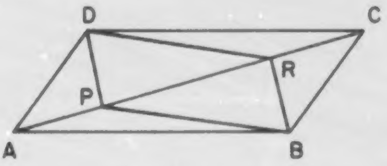
   j. It is equilateral and equiangular.

   k. Both pairs of its opposite angles are congruent.

   l. Each pair of its consecutive angles is supplementary.

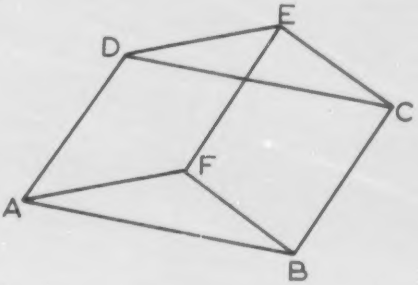
5. Given: ABCD is a parallelogram with diagonal $\overline{AC}$. $AP = RC$.

   Prove: DPBR is a parallelogram.



6. Given: Parallelograms AFED and FBCE, as shown in this plane figure.

   Prove: ABCD is a parallelogram.

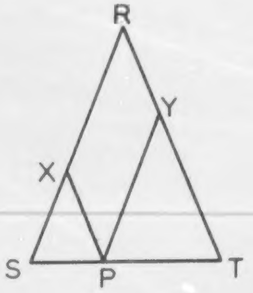


7. If lines are drawn parallel to the legs of an isosceles triangle through a point in the base of the triangle, then a parallelogram is formed and its perimeter is equal to the sum of the lengths of the legs.

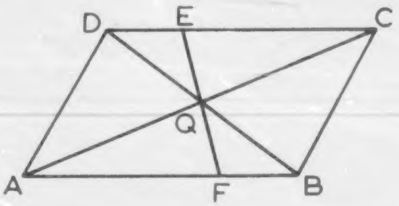
Given: In the figure $\overline{RS} \cong \overline{RT}$, $\overline{PX} \parallel \overline{RT}$, $\overline{PY} \parallel \overline{RX}$.

Prove: a. PXRY is a Parallelogram.

b. $PX + XR + RY + YP = RS + RT$.

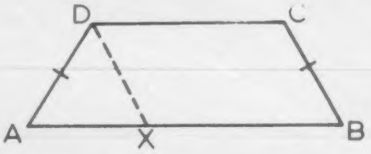


8. In this figure, if ABCD is a parallelogram with diagonals $\overline{AC}$ and $\overline{BD}$ intersecting in Q and $\overline{EF}$ is drawn through Q, prove that $\overline{EF}$ is bisected by Q.



9. Given the isosceles trapezoid ABCD in which $AD = CB$ and $\overline{CD} \parallel \overline{AB}$.

Prove $\angle A \cong \angle B$.



10. The median of a trapezoid is the segment joining the midpoints of its non-parallel sides.

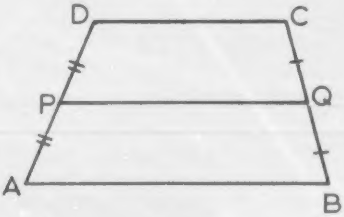
a. Prove the following theorem: The median of a trapezoid is parallel to the bases and equal in length to half the sum of the lengths of the bases.

Given: Trapezoid ABCD with $\overline{CD} \parallel \overline{AB}$, P the midpoint of $\overline{AD}$ and Q the midpoint of $\overline{BC}$.

Prove: $\overline{PQ} \parallel \overline{AB}$

$PQ = \frac{1}{2}(AB + CD)$.

(Hint: Draw $\overleftrightarrow{DQ}$ meeting $\overleftrightarrow{AB}$ at K.)



b. If AB = 9in. and DC = 7in, then PQ = ________.

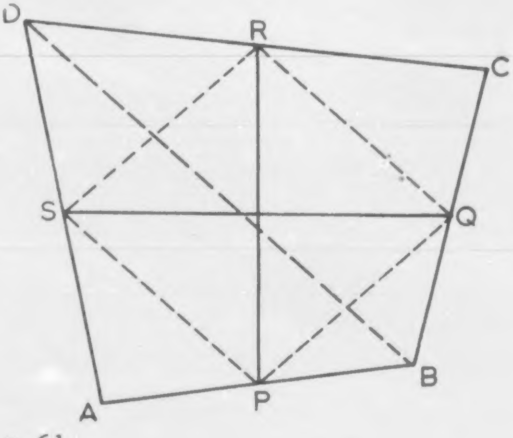
c. If $DC = 3\frac{1}{2}$ and $AB = 7$, then PQ = ________.

11. A convex quadrilateral with vertices labeled consecutively ABCD is called a kite if AB = BC and CD = DA. Sketch some kites. State as many theorems about a kite as you can and prove at least one of them.

12. Given: Quadrilateral ABCD with P, Q, R, S the midpoints of the sides.

Prove: RSPQ is a parallelogram, and $\overline{PR}$ and $\overline{SQ}$ bisect each other.

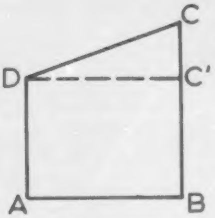
(Hint: Draw $\overline{RQ}$, $\overline{RS}$, $\overline{SP}$, $\overline{DB}$ and $\overline{PQ}$.)



[sec. 9-6]

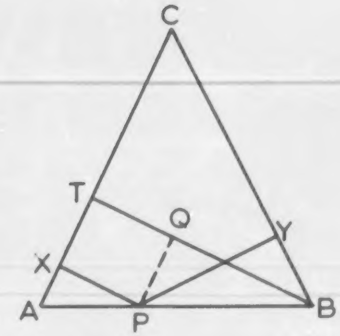
13. Given: In the figure $AD < BC$, $\overline{DA} \perp \overline{AB}$, $\overline{CB} \perp \overline{AB}$.

Prove: $m\angle C < m\angle D$.



*14. Prove that the sum of the lengths of the perpendiculars drawn from any point in the base of an isosceles triangle to the legs is equal to the length of the altitude upon either of the legs.

(Hint: Draw $\overline{PQ} \perp \overline{BT}$. Then the figure suggests that $\overline{PX}$ and $\overline{QT}$ are congruent, and that $\overline{PY}$ and $\overline{BQ}$ are congruent.)

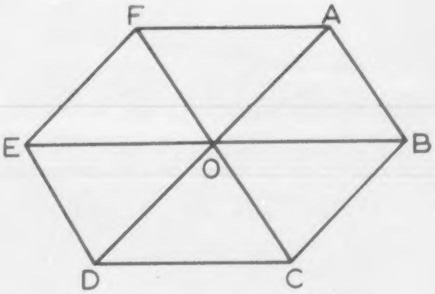


*15. Prove that the sum of the lengths of the perpendiculars drawn from any point in the interior of an equilateral triangle to the three sides is equal to the length of an altitude.

(Hint: Draw a segment, perpendicular to the altitude used, from the interior point.)

16. Given a hexagon as in the figure with $\overline{AB} \parallel \overline{OC}$, $\overline{BC} \parallel \overline{OD}$, $\overline{CD} \parallel \overline{OE}$, $\overline{DE} \parallel \overline{OF}$, $\overline{EF} \parallel \overline{OA}$.

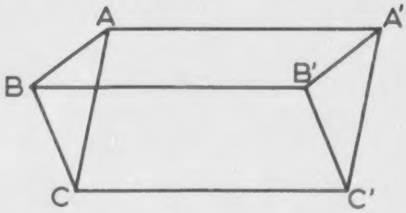
Prove: $\overline{FA} \parallel \overline{CD}$.



17. a. Given $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$ are parallel and $\overline{AB} \parallel \overline{A'B'}$, $\overline{BC} \parallel \overline{B'C'}$ as in figure.

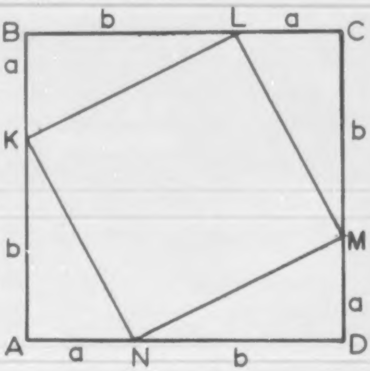
Prove: $\overline{AC} \parallel \overline{A'C'}$.

b. Is the figure necessarily a plane figure. Will your proof apply if it isn't?



18. Given ABCD is a square and the points K, L, M N divide the sides as shown, a and b being lengths of the indicated segments.

Prove: KLMN is a square.

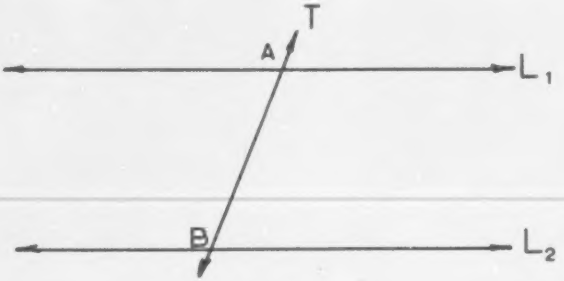


*19. Show that if ABCD is a parallelogram then D is in the interior of $\angle ABC$.

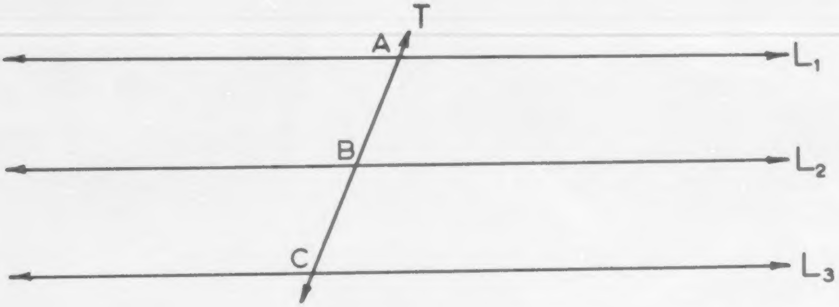
*20. Show that the diagonals of a parallelogram intersect each other.

9-7. Transversals To Many Parallel Lines.

Definitions: If a transversal intersects two lines L_1, L_2 in points A and B, then we say that L_1 and L_2 intercept the segment $\overline{AB}$ on the transversal.



Suppose that we have given three lines L_1, L_2, L_3 and a transversal intersecting them in points A, B and C. If AB = BC, then we say that the three lines intercept congruent segments on the transversal.



We shall prove the following:

Theorem 9-26. If three parallel lines intercept congruent segments on one transversal, then they intercept congruent segments on any other transversal.

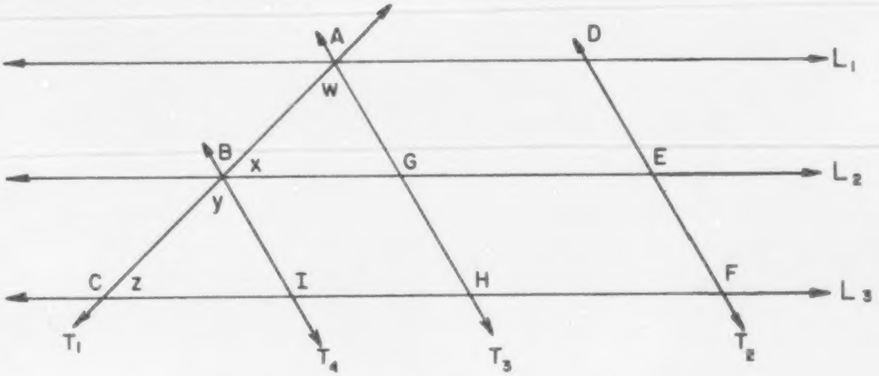
Proof: Let L_1, L_2 and L_3 be parallel lines, cut by a transversal T_1 in points A, B and C. Let T_2 be another transversal, cutting these lines in D, E, and F. We have given that

$$AB = BC;$$

and we need to prove that

$$DE = EF.$$

We will first prove the theorem for the case in which T_1 and T_2 are not parallel, and $A \neq D$, as in the figure:



Let T_3 be the line through A, parallel to T_2, intersecting L_2 and L_3 in G and H; and let T_4 be the line through B, parallel to T_2, intersecting L_3 in I. Let $\angle x$, $\angle y$, $\angle w$ and $\angle z$ be as indicated in the figure.

Statements	Reasons
1. $\angle x \cong \angle z$.	1. Theorem 9-9.
2. $AB = BC$.	2. Hypothesis.
3. $T_3 \parallel T_4$.	3. Theorem 9-11.
4. $\angle w \cong \angle y$.	4. Theorem 9-9.
5. $\triangle ABG \cong \triangle BCI$.	5. A. S. A.
6. $AG = BI$.	6. Definition of congruent triangles.
7. AGED and BIFE are parallelograms.	7. Definition of parallelograms.
8. $AG = DE$ and $BI = EF$.	8. Opposite sides of a parallelogram are congruent.
9. $DE = EF$.	9. Steps 6 and 8.

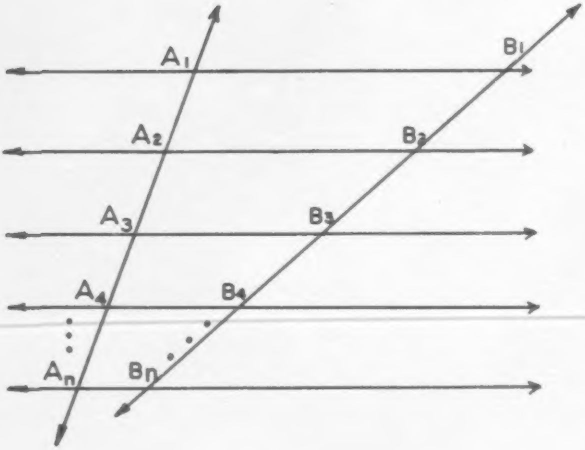
This proves the theorem for the case in which the two transversals are not parallel, and intersect L_1 in two different points. The other cases are easy.

(1) If the two transversals are parallel, like T_2 and T_3 in the figure, then the theorem holds, because opposite sides of a parallelogram are congruent. (Thus, if $AG = GH$, it follows that $DE = EF$.)

(2) If the two transversals intersect at A, like T_1 and T_3 in the figure, then the theorem holds; in fact, we have already proved that if $AB = BC$, then $AG = GH$.

The following corollary generalizes Theorem 9-26.

Corollary 9-26-1. If three or more parallel lines intercept congruent segments on one transversal, then they intercept congruent segments on any other transversal.



That is, given that

$$A_1A_2 = A_2A_3 = A_3A_4 = \ldots,$$

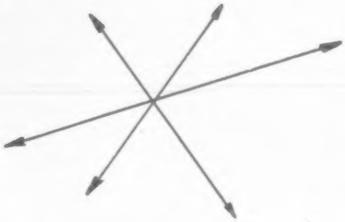
it follows that

$$B_1B_2 = B_2B_3 = B_3B_4 = \ldots,$$

and so on. This follows by repeated applications of the theorem that we have just proved.

Definition: Two or more sets are concurrent if there is a point which belongs to all of the sets.

In particular, three or more lines are concurrent if they all pass through one point.

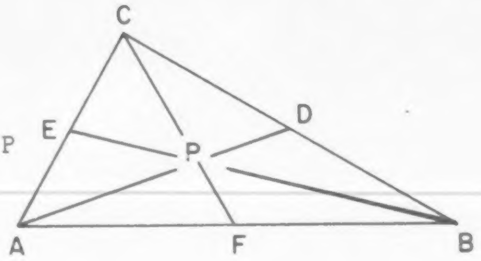


The following theorem is an interesting application of Corollary 9-26-1.

Theorem 9-27. The medians of a triangle are concurrent in a point two-thirds the way from any vertex to the mid-point of the opposite side.

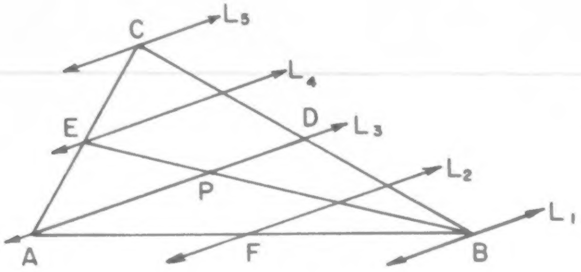
Given: In $\triangle ABC$, D, E and F are the mid-points of $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ respectively.

To Prove: There is a point P which lies on $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$; and $AP = \frac{2}{3} AD$, $BP = \frac{2}{3} BE$, $CP = \frac{2}{3} CF$.



Sketch of proof:

(1)

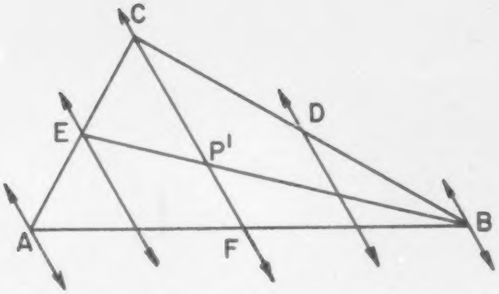


Let L_1, L_2, L_3, L_4 and L_5, with $L_3 = \overleftrightarrow{AD}$ be five parallel lines dividing $\overline{CB}$ into four congruent segments. Then

(a) L_3, L_4, L_5 divide $\overline{AC}$ into two congruent segments, and so E lies on L_4.

(b) L_1, L_2, L_3, L_4 divide $\overline{BE}$ into three congruent segments, and so if P is the point of intersection of $\overline{AD}$ and $\overline{BE}$, then $BP = \frac{2}{3} BE$.

(2)



In the same way, with lines parallel to $\overleftrightarrow{CF}$, we find that if P' is the intersection of $\overline{BE}$ and $\overline{CF}$, then $BP' = \frac{2}{3} BE$.

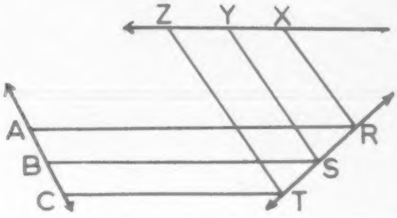
(3) From (1) and (2) and Theorem 2-4 it follows that $P' = P$, and therefore the three medians are concurrent.

(4) Since we now know that $\overline{CF}$ passes through P we can easily get $CP = \frac{2}{3} CF$ from the figure in (1), and similarly get $AP = \frac{2}{3} AD$ from the figure in (2).

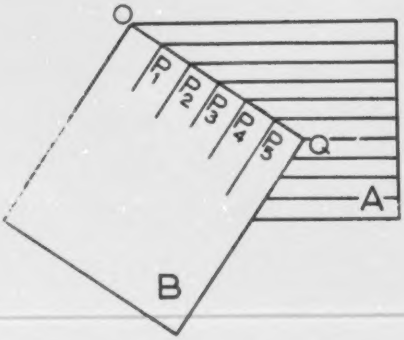
Definition: The centroid of a triangle is the point of concurrency of the medians.

Problem Set 9-7

1. Given: $AB = BC$.
   $\overline{AR} \parallel \overline{BS} \parallel \overline{CT}$.
   $\overline{RX} \parallel \overline{SY} \parallel \overline{TZ}$.
   a. Prove $ZY = YX$.
   b. Do $\overleftrightarrow{AC}$, $\overleftrightarrow{TR}$ and $\overleftrightarrow{ZX}$ have to be coplanar to carry out the proof?



2. The procedure at the right can be used to rule a sheet of paper, B, into columns of equal width. If A is an ordinary sheet of ruled paper and B is a second sheet placed over it as shown, explain why $OP_1 = P_1P_2 = P_2P_3 = P_3P_4$ $= P_4P_5 = P_5Q$.



3. Divide a given segment $\overline{AB}$ into five congruent parts by the following method:

   (1) Draw ray $\overrightarrow{AR}$ (not collinear with $\overline{AB}$.)

   (2) Use your ruler to mark off congruent segments $\overline{AN_1}$, $\overline{N_1N_2}$, $\overline{N_2N_3}$, $\overline{N_3N_4}$ and $\overline{N_4N_5}$ of any convenient length.

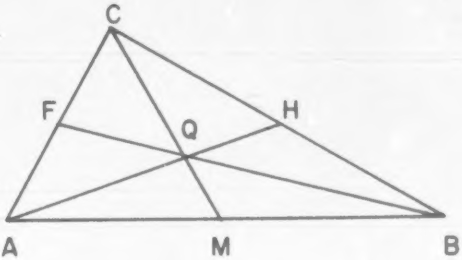
   (3) Draw $\overline{N_5B}$.

   (4) Measure $\angle AN_5B$ and use your protractor to draw corresponding angles congruent to $\angle AN_5B$ with vertices at N_4, N_3, N_2 and N_1.

   Explain why $\overline{AB}$ is divided into congruent parts.

4. The medians of $\triangle ABC$ meet at Q, as shown in this figure.

   If $BF = 18$, $AQ = 10$, $CM = 9$, then $BQ =$ _____, $QH =$ _______, $CQ =$ _____.

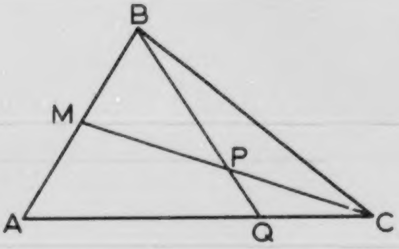


5. In equilateral $\triangle ABC$ if one median is 15 inches long, what is the distance from the centroid to A? To the midpoint of $\overline{AB}$? To side $\overline{AC}$?

*6. Given: $\overline{CM}$ bisects $\overline{AB}$ at M. $\overline{BQ}$ bisects $\overline{CM}$ at P.

Prove: Q is a trisection point of $\overline{AC}$; that is, $AQ = 2QC$.

(Hint: On the ray opposite to $\overrightarrow{CB}$ take point E such that $CE = CB$ and show that $\overline{BQ}$ is contained in a median of $\triangle ABE$.)



*7. What is the smallest number of congruent segments into which $\overline{AC}$ can be divided by some set of *equally spaced* parallels which will include the parallels $\overleftrightarrow{AR}$, $\overleftrightarrow{BS}$ and $\overleftrightarrow{CT}$ if:

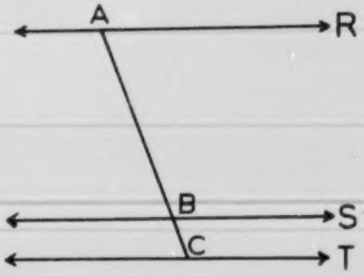
a. $AB = 2$ and $BC = 1$?

b. $AB = 1\frac{1}{3}$ and $BC = 1$?

c. $AB = 21$ and $BC = 6$?

d. $AB = 1.414$ and $BC = 1$?

e. $AB = \sqrt{2}$ and $BC = 1$?

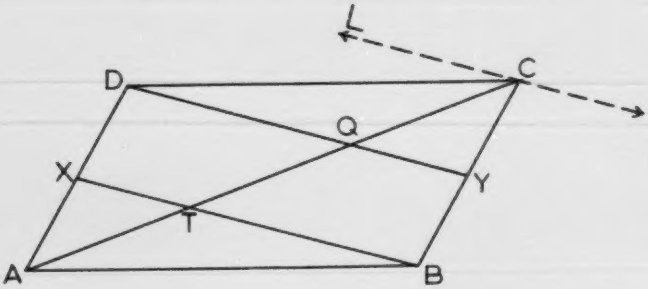


*8. Prove that the lines through opposite vertices of a parallelogram and the midpoints of the opposite sides trisect a diagonal.

(Hint: Through an extremity of the diagonal, consider a parallel to one of the lines.)

Given: ABCD is a parallelogram.
X and Y are midpoints.

Prove: AT = TQ = QC.



Review Problems

1. Indicate whether each of the following statements is true in ALL cases, true in SOME cases and false in others, or true in NO case, using the letter A, S or N:

   a. Line segments in the same plane which have no point in common are parallel.

   b. If two sides of a quadrilateral ABCD are parallel, then ABCD is a trapezoid.

   c. Two angles in a plane which have their sides respectively perpendicular are congruent.

   d. If two parallel lines are cut by a transversal, then a pair of alternate exterior angles are congruent.

e. If two lines are cut by a transversal, then the rays bisecting a pair of alternate interior angles are parallel.

f. In a plane, if a line is parallel to one of two parallel lines, it is parallel to the other.

g. In a plane two lines are either parallel or they intersect.

h. In a parallelogram the opposite angles are supplementary.

i. The diagonals of a rhombus bisect each other.

j. All three exterior angles of a triangle are acute.

k. A quadrilateral having two opposite angles which are right angles is a rectangle.

l. The diagonals of a rhombus are congruent.

m. If a quadrilateral is equilateral, then all of its angles are congruent.

n. If two opposite sides of a quadrilateral are congruent and the other two sides are parallel, the quadrilateral is a parallelogram.

o. The diagonals of a rhombus bisect the angles of the rhombus.

p. If the diagonals of a parallelogram are perpendicular, the parallelogram is a square.

q. If a median to one side of a triangle is not an altitude, the other two sides are unequal in length.

r. Either diagonal of a parallelogram makes two congruent triangles with the sides.

s. If a diagonal of a quadrilateral divides it into two congruent triangles, the quadrilateral is a parallelogram

t. If two lines are intersected by a transversal, the alternate interior angles are congruent.

u. All four sides of a rectangle are congruent.

v. All four angles of a rhombus are congruent.

w. A square is a rhombus.

x. A square is a rectangle.

2. Would the following information about a quadrilateral be sufficient to prove it a parallelogram? A square? A rhombus? A rectangle? Consider each item of information separately.

a. Its diagonals bisect each other.

b. Its diagonals are congruent.

c. It is equilateral.

d. It is equilateral and equiangular.

e. A diagonal bisects two angles.

f. Every two opposite sides are congruent.

g. Some two consecutive sides are congruent and perpendicular.

h. The diagonals are perpendicular.

i. Every two opposite angles are congruent.

j. Each diagonal bisects two angles.

k. Every two consecutive angles are supplementary.

l. Every two consecutive sides are congruent.

3. $\angle A$ and $\angle B$ have their sides respectively parallel.

a. If only one pair of corresponding sides extend in the same direction the angles are ______________.

b. If corresponding sides extend in opposite directions, then the angles are ______________.

In Problems 4, 5 and 6 below select the one word or phrase that makes the statement true.

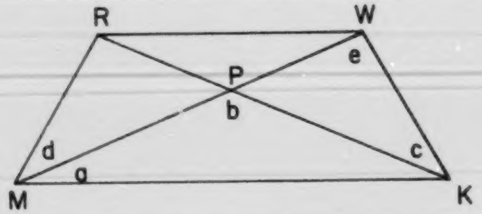
4. The bisectors of the opposite angles of a non-equilateral parallelogram (a) coincide, (b) are perpendicular, (c) intersect but are not perpendicular, (d) are parallel.

5. The figure formed by joining the consecutive mid-points of the sides of a rhombus is (a) a rhombus, (b) a rectangle, (c) a square, (d) none of these answers.

*6. The figure formed by joining the consecutive mid-points of the sides of quadrilateral ABCD is a square (a) if, and only if, the diagonals of ABCD are congruent and perpendicular, (b) if, and only if, the diagonals of ABCD are congruent, (c) if, and only if, ABCD is a square, (d) if, and only if, the diagonals of ABCD are perpendicular.

7. In the left-hand column below, certain conditions are specified. In the right-hand column, some deducible conclusions are left for you to complete.

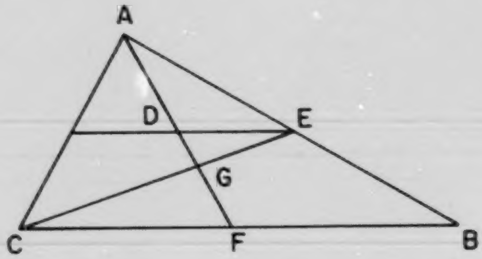
Given: $\overline{MW}$ and $\overline{KR}$ are diagonals of MKWR.



	Conditions:	Conclusions:
a.	MKWR is a parallelogram, $m\angle a = 30$, and $m\angle WKM = 110$.	$m\angle d$ = _____ and $m\angle RWK$ = _____
b.	MKWR is a rectangle and $m\angle a = 30$.	$m\angle d$ = _____ and $m\angle b$ = _____
c.	MKWR is a rhombus, $m\angle a = 30$ and MK = 6.	$m\angle b$ = _____ and **RK** = _____

8. Given: In the figure
AE = EB, GF = 8,
CF = FB, $\overleftrightarrow{DE} \parallel \overleftrightarrow{CB}$.

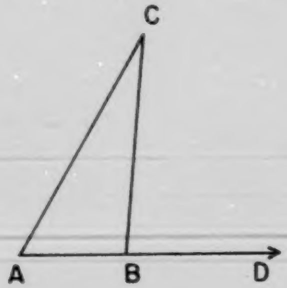
Find: DG.



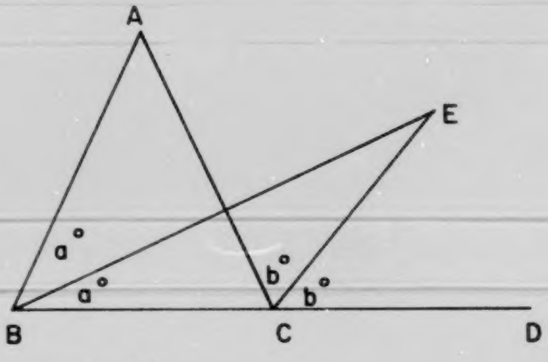
9. If the perimeter (sum of lengths of sides) of a triangle is 18 inches, what is the perimeter of the triangle formed by joining the mid-points of sides of the first triangle?

10. a. If $m\angle A = 30$ and $m\angle C = 25$, what is the measure of $\angle CBD$?

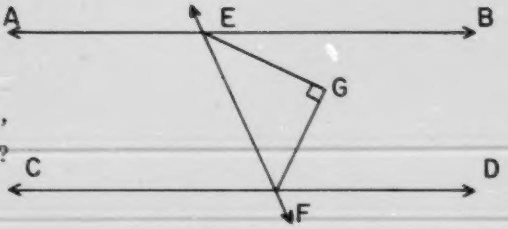
b. If $m\angle A = a$ and $m\angle C = \frac{a}{2}$, what is $m\angle CBD$? $m\angle ABC$?



11. Show that the measure of $\angle E$, formed by the bisector of $\angle ABC$ and the bisector of exterior $\angle ACD$ of $\triangle ABC$, is equal to $\frac{1}{2} m\angle A$.

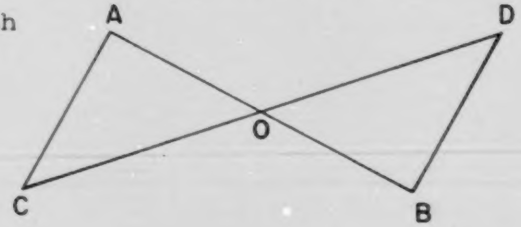


12. In the figure $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $\overrightarrow{EG}$ bisects $\angle$ BEF, m$\angle$ G = 90. If the measure of $\angle$ GEF = 25, what is the measure of $\angle$ GFD?



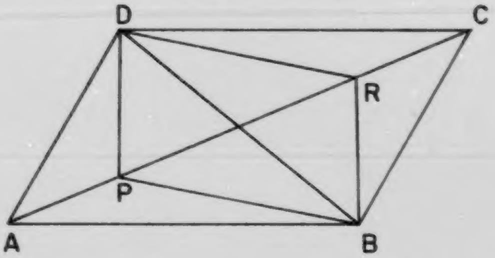
13. Given: $\overline{AB}$ and $\overline{CD}$ which bisect each other at O.

Prove: $\overline{AC} \parallel \overline{BD}$.



14. Given: ABCD is a parallelogram with diagonals $\overline{AC}$ and $\overline{DB}$. AP = RC < $\frac{1}{2}$ AC.

Prove: DPBR is a parallelogram.



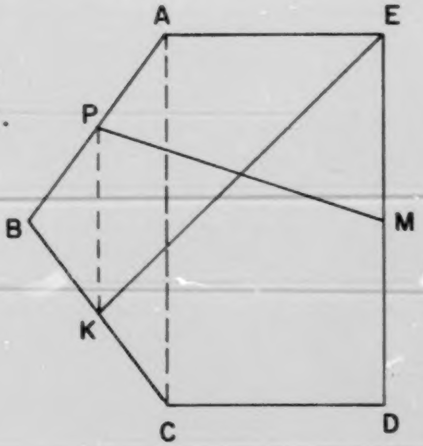
15. Prove or disprove:

If a quadrilateral has one pair of parallel sides and one pair of congruent sides, then the quadrilateral is a parallelogram.

*16. In $\triangle$ABC, median $\overline{AM}$ is congruent to $\overline{MC}$. Prove that $\triangle$ABC is a right triangle.

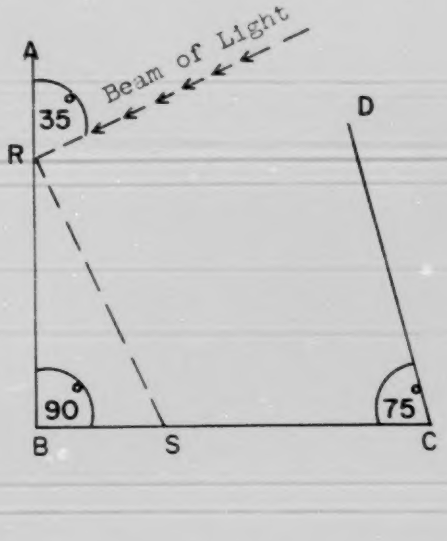
17. Prove: If the bisectors of two consecutive angles of a parallelogram intersect, they are perpendicular to each other.

*18. Given: ABCDE is a pentagon as shown. $\overleftrightarrow{AE} \parallel \overleftrightarrow{CD}$. AE = CD.
P is mid-point of $\overline{AB}$.
K is mid-point of $\overline{BC}$.
EM = $\frac{1}{2}$ ED.
Prove: $\overline{KE}$ bisects $\overline{PM}$.

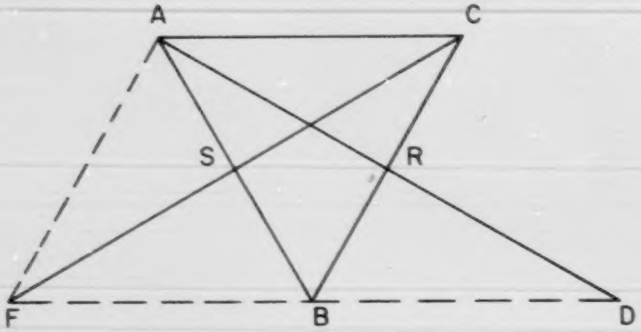


19. When a beam of light is reflected from a smooth surface, the angle between the incoming beam and the surface is congruent to the angle between the reflected beam and the surface.

In the accompanying figure, m∠ ABC = 90, m ∠ BCD = 75, and the beam of light makes an angle of 35° with $\overline{AB}$. Copy the figure and complete the path of the light beam as it reflects from $\overline{AB}$, from $\overline{BC}$, from $\overline{DC}$, and from $\overline{AB}$ again. At what angle does the beam reflect from $\overline{AB}$ the second time?



20. Given triangle ABC with $\overline{AR}$ and $\overline{CS}$ medians. If $\overline{AR}$ is extended its own length to D, and $\overline{CS}$ is extended its own length to F, prove that F, B and D as shown are collinear.



Chapter 10

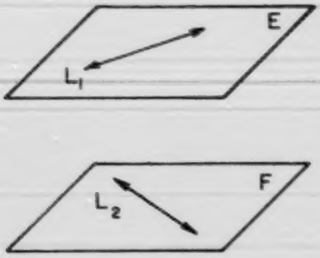
PARALLELS IN SPACE

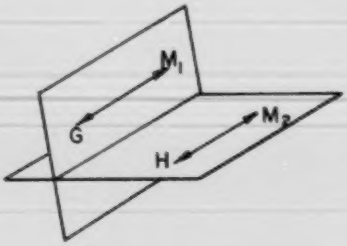
10-1. *Parallel Planes*.

Definition: Two planes, or a plane and a line, are *parallel* if they do not intersect.

If planes E_1 and E_2 are parallel we write $E_1 \parallel E_2$; if line L and plane E are parallel we write $L \parallel E$ or $E \parallel L$. As we will soon see, parallels in space behave in somewhat the same way as parallel lines in a plane. To study them we do not need any new postulates.

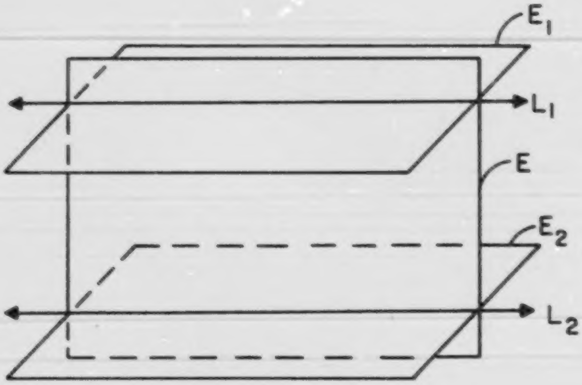
However, in spite of the similarities it is necessary, in studying theorems and their proofs in this chapter, to distinguish carefully between parallel *lines* and parallel *planes*. Two parallel planes such as E and F in the first Figure below contain lines such as L_1 and L_2 which are *not* parallel. And the second Figure shows parallel lines M_1 and M_2 lying in *intersecting* planes G and H.





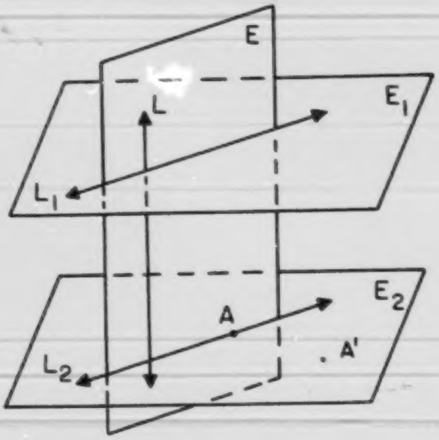
The following theorem describes a common situation in which parallel planes and parallel lines occur in the same Figure.

Theorem 10-1. If a plane intersects two parallel planes, then it intersects them in two parallel lines.



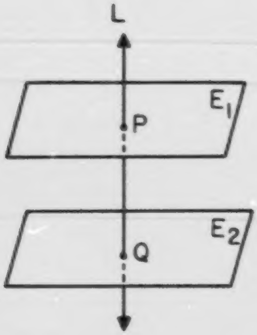
Proof: Given a plane E, intersecting two parallel planes E_1 and E_2. By Postulate 8, the intersections are lines L_1 and L_2. These lines are in the same plane E; and they have no point in common because E_1 and E_2 have no point in common. Therefore, they are parallel by the definition of parallel lines.

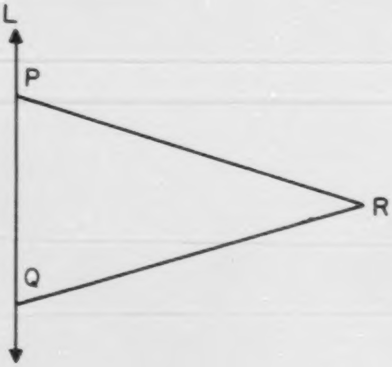
Theorem 10-2. If a line is perpendicular to one of two parallel planes, it is perpendicular to the other.



Proof: Let planes E_1 and E_2 be parallel and let line L be perpendicular to E_1. In E_2 take a point A not on L, and let E be the plane determined by L and A. By the preceding theorem E intersects E_1 and E_2 in parallel lines L_1 and L_2. $L \perp L_1$ since $L \perp E_1$, and so by Theorem 9-12 (look it up) $L \perp L_2$. Now take a point A' in E_2 not on L_2 and repeat the process. We thus obtain two lines in E_2 each perpendicular to L, and so $L \perp E_2$, by Theorem 8-3.

Theorem 10-3. Two planes perpendicular to the same line are parallel.





Proof: The figure on the left shows what happens when $E_1 \perp L$ at P and $E_2 \perp L$ at Q: we wish to show $E_1 \parallel E_2$. If E_1 and E_2 are not parallel, they intersect. Let R be a common point. Consider the lines $\overleftrightarrow{PR}$ and $\overleftrightarrow{QR}$. Then $L \perp \overleftrightarrow{PR}$ and $L \perp \overleftrightarrow{QR}$ because L is perpendicular to every line in E_1 through P and every line in E_2 through Q. This gives two perpendiculars to a line from an external point, which is impossible, by Theorem 6-3.

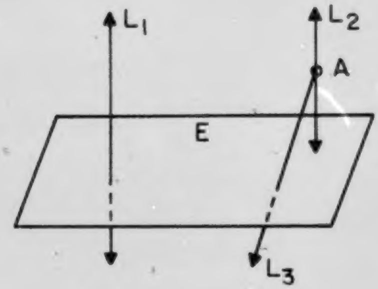
Corollary 10-3-1. If two planes are each parallel to a third plane, they are parallel to each other.

Proof: Let $E_1 \parallel E_3$, $E_2 \parallel E_3$. Let L be a line perpendicular to E_3. By Theorem 10-2 $L \perp E_1$ and $L \perp E_2$. Thus E_1 and E_2 are each perpendicular to L and $E_1 \parallel E_2$ by the Theorem 10-3.

Theorem 10-4. Two lines perpendicular to the same plane are parallel.

Proof: By Theorem 8-8 two such lines are coplanar. Since they are perpendicular to the given plane, say at points A and B, they are perpendicular to $\overleftrightarrow{AB}$. Hence by Theorem 9-2 they are parallel.

Corollary 10-4-1. A plane perpendicular to one of two parallel lines is perpendicular to the other.

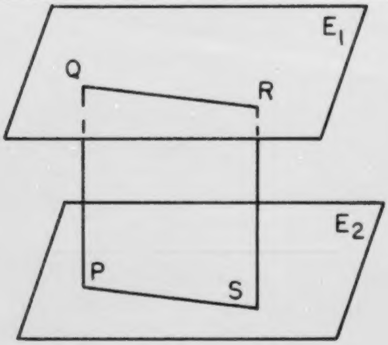


Proof: Let $L_1 \parallel L_2$, $L_1 \perp E$. Let L_3 be a line perpendicular to E through any point A of L_2. L_3 exists by Theorem 8-9. Then by Theorem 10-4 $L_1 \parallel L_3$. Hence, by the Parallel Postulate $L_3 = L_2$, and so $L_2 \perp E$.

Corollary 10-4-2. If two lines are each parallel to a third they are parallel to each other.

Proof: Let $L_1 \parallel L_2$, $L_1 \parallel L_3$. Let E be a plane perpendicular to L_1. By the above corollary $E \perp L_2$ and $E \perp L_3$, and so by the above theorem $L_2 \parallel L_3$.

Theorem 10-5. Two parallel planes are everywhere equidistant. That is, all segments perpendicular to the two planes and having their end points in the planes have the same length.



Proof: Let $\overline{PQ}$ and $\overline{RS}$ be perpendicular segments between the parallel planes E_1 and E_2. By Theorem 10-2, each of the segments is perpendicular to each of the planes. By Theorem 10-4, $\overleftrightarrow{PQ} \parallel \overleftrightarrow{RS}$; and this means, in particular, that $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ lie in the same plane E_3. By Theorem 10-1, $\overleftrightarrow{QR} \parallel \overleftrightarrow{PS}$. Therefore, PQRS is a parallelogram. Opposite sides of a parallelogram are congruent. Therefore, $PQ = RS$, which was to be proved. (Obviously PQRS is a rectangle, but this fact does not need to be mentioned in the proof.)

Problem Set 10-1

1. Draw a small sketch to illustrate the hypothesis of each of the following statements. Below each sketch indicate whether the statement is true or false.

    a. If a line is perpendicular to one of two parallel planes it is perpendicular to the other.

    b. Two lines parallel to the same plane may be perpendicular to each other.

    c. Two planes perpendicular to the same line may intersect.

    d. If a plane intersects two intersecting planes, the lines of intersection may be parallel.

    e. If two planes are both perpendicular to each of two parallel lines, the segments of the two lines intercepted between the planes are congruent.

    f. If two planes, perpendicular to the same line, are intersected by a third plane, the lines of intersection are parallel.

    g. If a line lies in a plane, a perpendicular to the line is perpendicular to the plane.

    h. If a line lies in a plane, a perpendicular to the plane at some point of the line is perpendicular to the line.

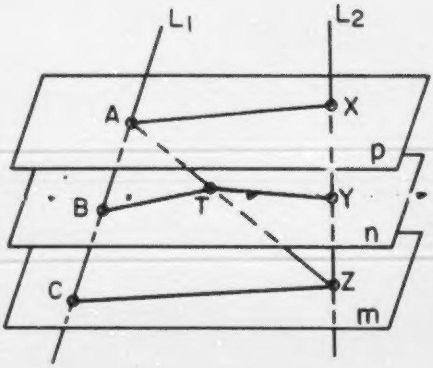
    i. If two lines are parallel, every plane containing only one of them is parallel to the other line.

    j. If two lines are parallel, every line intersecting one of them intersects the other.

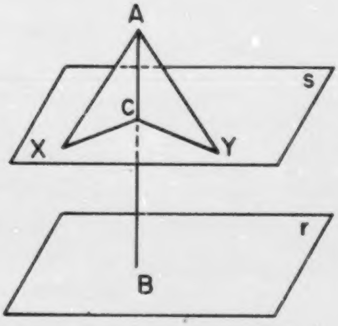
    k. If two planes are parallel, any line in one of them is parallel to the other.

    l. If two planes are parallel, any line in one of them is parallel to any line in the other.

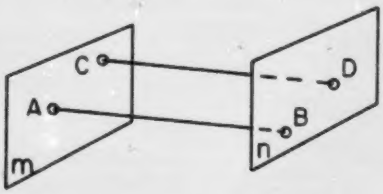
2. Given lines L_1 and L_2 intersecting parallel planes m, n, and p at points A, B, C, and X, Y, Z, with B the mid-point of $\overline{AC}$. Prove: XY = YZ.



3. Given: plane s $\parallel$ plane r, $\overline{AB} \perp$ r. CX = CY in plane s. Prove: AX = AY.

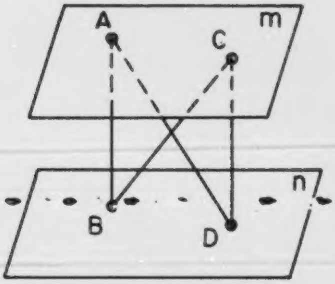


4. Given: A, C in m; B, D in n, m $\perp \overline{AB}$, n $\perp \overline{AB}$, m $\perp \overline{CD}$. Prove: n $\perp \overline{CD}$.

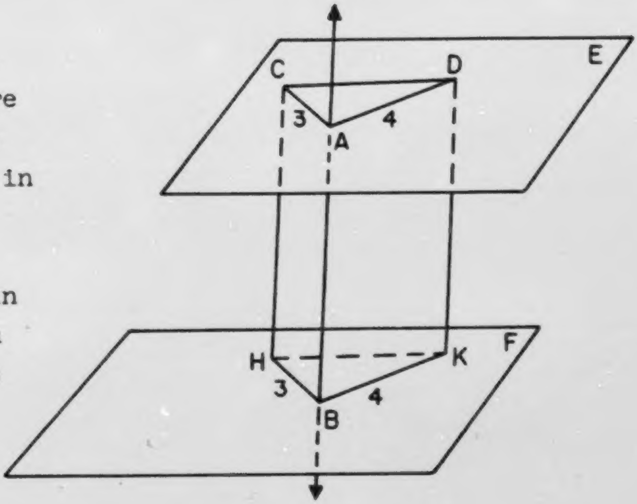


5. Given: In the figure $m \parallel n$, $\overline{AB} \perp n$, $\overline{CD} \perp n$.

Prove: $AD = CB$.



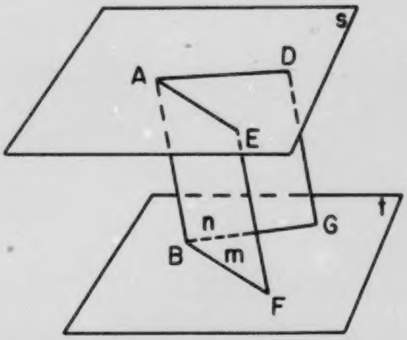
*6. Planes E and F are perpendicular to $\overleftrightarrow{AB}$. Lines $\overleftrightarrow{BK}$ and $\overleftrightarrow{BH}$, in plane F, determine with $\overleftrightarrow{AB}$ two planes which intersect E in $\overleftrightarrow{AD}$ and $\overleftrightarrow{AC}$. Certain lengths are given, as in the figure.



Are BKDA and BACH parallelograms? Can you give a further description of them? Is $\triangle BHK \cong \triangle CAD$? Can you give the length of $\overline{CD}$?

*7. In the figure half planes n and m have a common edge $\overleftrightarrow{AB}$ and intersect parallel planes s and t in lines $\overleftrightarrow{AD}$, $\overleftrightarrow{AE}$, $\overleftrightarrow{BG}$, and $\overleftrightarrow{BF}$ as shown.

Prove that $\angle DAE \cong \angle GBF$.

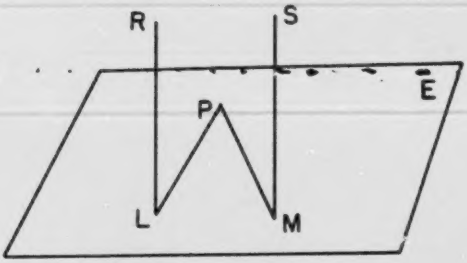


*8. Show how to determine a plane containing one of two skew lines and parallel to the other. Prove your construction.

*9. Given: $\overline{PL}$ and $\overline{PM}$ lie in plane E. $\overline{RL} \perp \overline{LP}$, $\overline{SM} \perp \overline{MP}$. $\overline{RL} \parallel \overline{SM}$.

Prove: $\overline{RL} \perp E$, $\overline{SM} \perp E$.

(Hint: At P draw $\overline{QP} \parallel \overline{RL}$ at P.)

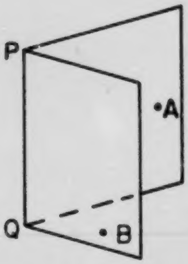


10-2. Dihedral Angles, Perpendicular Planes.

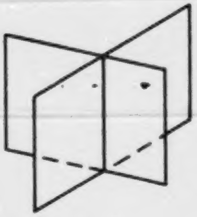
We have considered perpendicularity between two lines, and between a line and a plane. We have yet to define perpendicularity between two planes. This can be done in various ways, and we choose the one that has the closest analogy with the definition of perpendicular lines.

Definitions: A dihedral angle is the union of a line and two non-coplanar half-planes having this line as their common edge. (Compare with the definition of angle in Chapter 4.) The line is called the edge of the dihedral angle. The union of the edge and either half-plane is called a face, or side, of the dihedral angle.

If $\overleftrightarrow{PQ}$ is the edge, and A and B points on different sides, we denote the dihedral angle by $\angle$A-PQ-B.

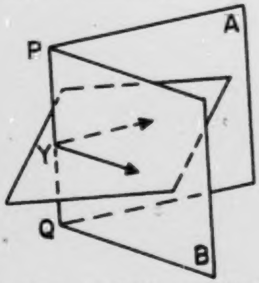


Analogous to the discussion on page 88 we see that two intersecting planes determine four dihedral angles.



Terms such as vertical, interior, exterior, etc. can be applied to dihedral angles. Definitions of these terms can be considered an exercise for the student.

To define right dihedral angles, however, we need to talk about the measure of a dihedral angle. One might at first think that we must introduce four new postulates, analogous to those in Section 4-3. However, this is not necessary, for we can relate each dihedral angle with an ordinary angle, as follows:



Definition: Through any point on the edge of the dihedral angle pass a plane perpendicular to the edge, intersecting each of the sides in a ray. The angle formed by these rays is called a plane angle of the dihedral angle.

The sides of the plane angle are perpendicular to the edge of the dihedral angle, so another way of defining the plane angle would be the angle formed by two rays, one in each side of the dihedral angle, and perpendicular to its edge at the same point.

It is natural at this point to use the measure of the plane angle as a measure of the dihedral angle, but before we do this we must prove that any two plane angles of a dihedral angle have the same measure.

Theorem 10-5. Any two plane angles of a given dihedral angle are congruent.

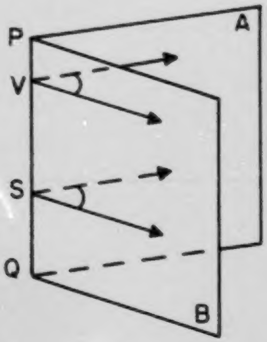


Figure A.

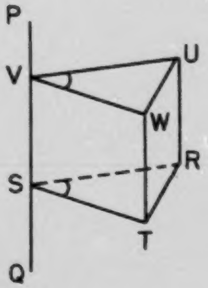


Figure B.

Proof: Let V and S be the vertices of two plane angles of $\angle$A-PQ-B. (Figure A.) On the sides of $\angle$V take points U and W distinct from V. On the sides of $\angle$S take points R and T such that SR = VU, ST = VW. (Figure B.) $\overline{VU}$ and $\overline{SR}$ are coplanar and perpendicular to $\overline{PQ}$; hence they are parallel by Theorem 9-2. Hence by Theorem 9-20 (look it up) VURS is a parallelogram and UR = VS and $\overline{UR} \parallel \overline{VS}$. Similarly, WT = VS and $\overline{WT} \parallel \overline{VS}$. Hence UR = WT and $\overline{UR} \parallel \overline{WT}$, the latter fact following from Corollary 10-4-2. URTW is thus a parallelogram, and UW = RT. It follows from the S.S.S. Theorem that $\Delta UVW \cong \Delta RST$, and so $m\angle UVW = m\angle RST$.

Thus we can make the following definitions.

Definitions: The measure of a dihedral angle is the real number which is the measure of any of its plane angles. A dihedral angle is a right dihedral angle if its plane angles are right angles. Two planes are perpendicular if they determine right dihedral angles.

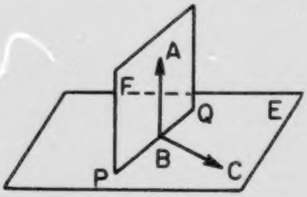
The following are some immediate consequences of these definitions. Their proofs are left as exercises.

Corollary 10-6-1. If a line is perpendicular to a plane, then any plane containing this line is perpendicular to the given plane.

Given: $\overleftrightarrow{AB} \perp E$, F contains $\overleftrightarrow{AB}$.

Prove: $F \perp E$.

(Hint: Take $\overleftrightarrow{BC} \perp \overleftrightarrow{PQ}$ in E.)

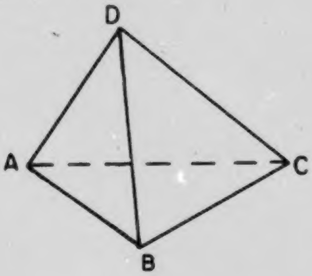


Corollary 10-6-2. If two planes are perpendicular, then any line in one of them perpendicular to their line of intersection, is perpendicular to the other plane.

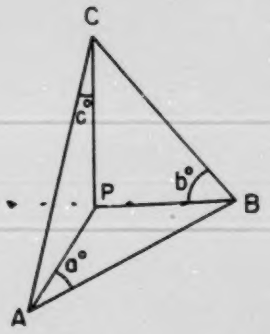
(Hint: In the above figure; given $F \perp E$, $\overleftrightarrow{AB} \perp \overleftrightarrow{PQ}$; prove $\overleftrightarrow{AB} \perp E$. Take $\overleftrightarrow{BC}$ as before.)

Problem Set 10-2

1. Name the six dihedral angles in this three dimensional figure.



2. Each of $\overline{AP}$, $\overline{BP}$ and $\overline{CP}$ is perpendicular to the other two. $a = b = c = 45$. What is the measure of $\angle$ C-PA-B? of $\angle$ CAB?



3. Draw a small sketch to illustrate the hypothesis of each of the following statements. Then indicate whether each is True (1) or False (0).

   a. If a plane and a line not in it are both perpendicular to the same line, they are parallel to each other.

   b. If a plane and a line not in it are both parallel to the same line they are parallel to each other.

   c. If parallel planes E and F are cut by plane Q, the lines of intersection are perpendicular.

   d. If two planes are parallel to the same line they are parallel to each other.

   e. Two lines parallel to the same plane are parallel to each other.

   f. Segments of parallel lines intercepted between two parallel planes are congruent.

   g. If planes E and F are perpendicular to $\overleftrightarrow{AB}$, then they intersect in line $\overleftrightarrow{HQ}$.

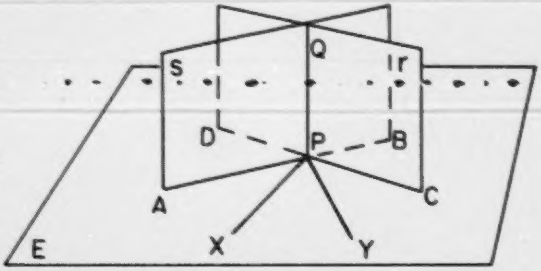
   h. Two planes perpendicular to the same plane are parallel to each other.

   i. Two lines perpendicular to the same line at the same point are perpendicular to each other.

   j. A plane perpendicular to one of two intersecting planes must intersect the other.

k. If two intersecting planes are each perpendicular to a third plane, their line of intersection is perpendicular to the third plane.

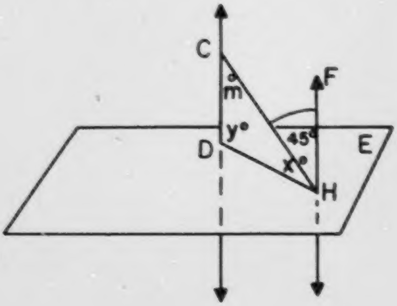
4. Prove: If two intersecting planes are each perpendicular to a third plane, their intersection is perpendicular to that third plane.



Given: Planes r and s intersect in PQ (P being chosen for convenience on plane E). r $\perp$ E and s $\perp$ E.

Prove: $\overline{QP} \perp E$. (Hint: In plane E, draw $\overline{XP} \perp \overline{DC}$ and $\overline{YP} \perp \overline{AB}$, and use Corollary 10-6-2.)

5. $\overleftrightarrow{CD}$ and $\overleftrightarrow{FH}$ are perpendicular to plane E. Other given information is as shown in the figure.



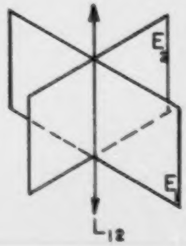
x = __?__; m = __?__;

y = __?__.

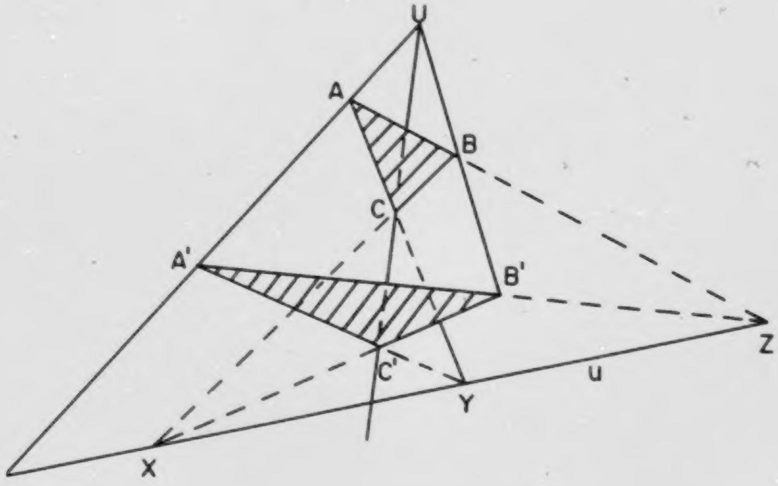
Which two segments have the same length?

*6. Prove the following theorem: If three planes E_1, E_2 and E_3 intersect in pairs and determine three lines L_{12}, L_{13} and L_{23}, then either the three lines are concurrent or each pair of the lines are parallel.

(Hint: The figure shows E_1 and E_2 meeting in L_{12}. If $E_3 \parallel L_{12}$ will the three lines L_{12}, L_{13} and L_{23}, be concurrent or parallel? Give proof. If E_3 intersects L_{12} in some point P will the three lines be concurrent or parallel? Give proof.)



*7. Desargues' Theorem. If two triangles lying in non-parallel planes are such that the lines joining corresponding vertices are concurrent, then if corresponding side-lines intersect, their points of intersection are collinear.



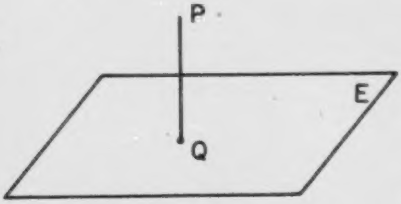
Restatement. Given the triangles $\triangle ABC$ and $\triangle A'B'C'$ in non-parallel planes such that $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CC'}$ intersect at U. Let the lines $\overleftrightarrow{CB}$ and $\overleftrightarrow{C'B'}$ meet at X, $\overleftrightarrow{CA}$ and $\overleftrightarrow{C'A'}$ meet at Y, and $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ meet at Z. Prove that the points X, Y, Z lie on a line.

10-3. Projections.

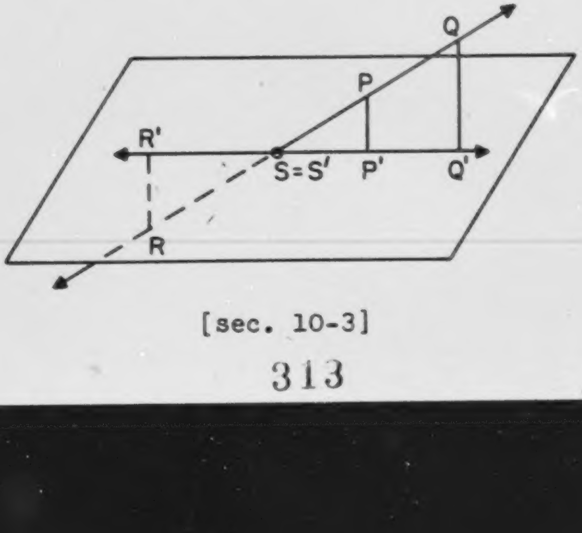
You are familiar with a slide projector which projects each point of a slide onto a screen. Each figure in the slide is projected as an enlarged figure on the screen. In this section you will notice certain differences and certain similarities between this familiar kind of projection and the kind of geometric projection which is presented.

Definition: The projection of a point into a plane is the foot of the perpendicular from the point to the plane. (By Theorem 8-10 this perpendicular exists and is unique.)

In the figure, Q is the projection of P into E.

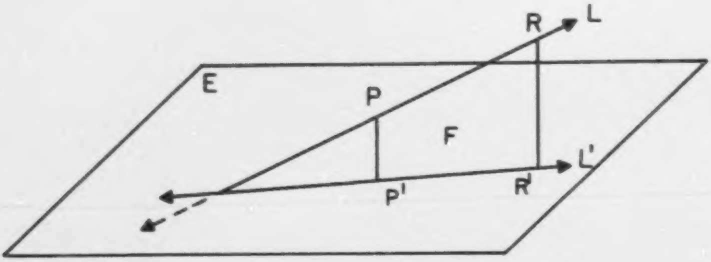


Definition: The projection of a line into a plane is the set of points which are projections into the plane of the points of the line.



In the figure, P' is the projection of P, Q' is the projection of Q, and so on. It looks as if the projection of the line is a line; and in fact this is what always happens, except when the line and the plane are perpendicular.

Theorem 10-7. The projection of a line into a plane is a line, unless the line and the plane are perpendicular.



Proof: Let L be a line not perpendicular to plane E.

Case 1. L lies in E. Then each point of L lies in E and is its own projection. (That is, a line through such a point P, perpendicular to E, intersects E in P.) Thus, the projection of L is just L itself, and so is certainly a line.

Case 2. L does not lie in E. Let P be a point of L that is not in E, let P' be the projection of P into E, and let F be the plane determined by the intersecting lines L and $\overleftrightarrow{PP'}$. F and E have point P' in common, and so, they intersect in a line which we call L'. (Postulate 8.) We want to show that L' is the projection of L.

To do this we must show two things:

(1) If R is a point of L, then its projection is a point of L'. This will show that the projection of L lies on L', but it will not assure us that the projection of L constitutes all of L'. To show the latter we must prove

(2) If S' is any point of L' there is a point S of L whose projection is S'.

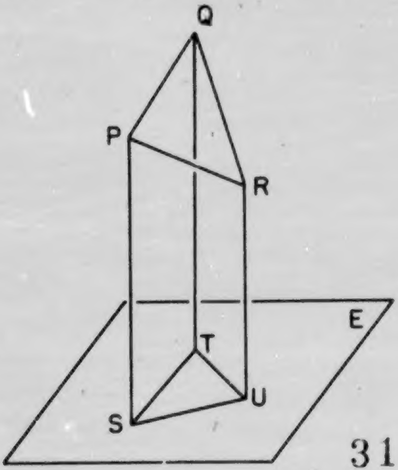
We can prove these two parts of Case 2 as follows:

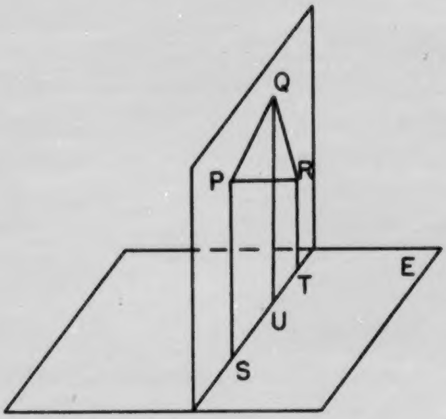
Proof of (1): If R = P, then R' = P' and so R' lies on L'. So suppose R is different from P. Then $\overleftrightarrow{PP'}$ and $\overleftrightarrow{RR'}$ are coplanar, by Theorem 8-8. Since F is the only plane containing P, R and P' (Postulate 7.), R' is in F. R' is also in E. Therefore R' is on L', since L', being the intersection of E and F, contains all points common to E and F.

Proof of (2): If S' is any point of L', then the line M through S' perpendicular to E is coplanar with $\overleftrightarrow{PP'}$ (or coincides with it if S' = P') and so lies in F. Therefore M intersects L (why?) at some point S. S' is the projection of S. This completes the proof of Theorem 10-7.

If a line is perpendicular to a plane its projection into the plane is a single point.

The idea of projection can be defined more generally, for any set of points. If A is any set of points, then the projection of A into the plane E is simply the set of all projections of points of A. For example, the projection of a triangle is usually a triangle, although in certain exceptional cases it may be a segment.



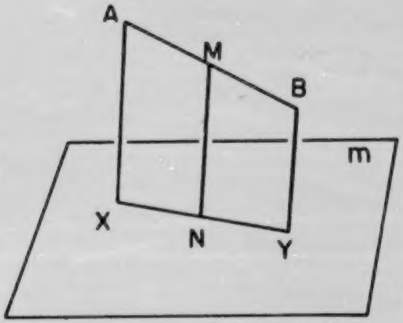


On the left, the projection of $\triangle PQR$ is $\triangle STU$. On the right, the plane that contains $\triangle PQR$ is perpendicular to E, so that the projection of $\triangle PQR$ is simply the segment $\overline{ST}$.

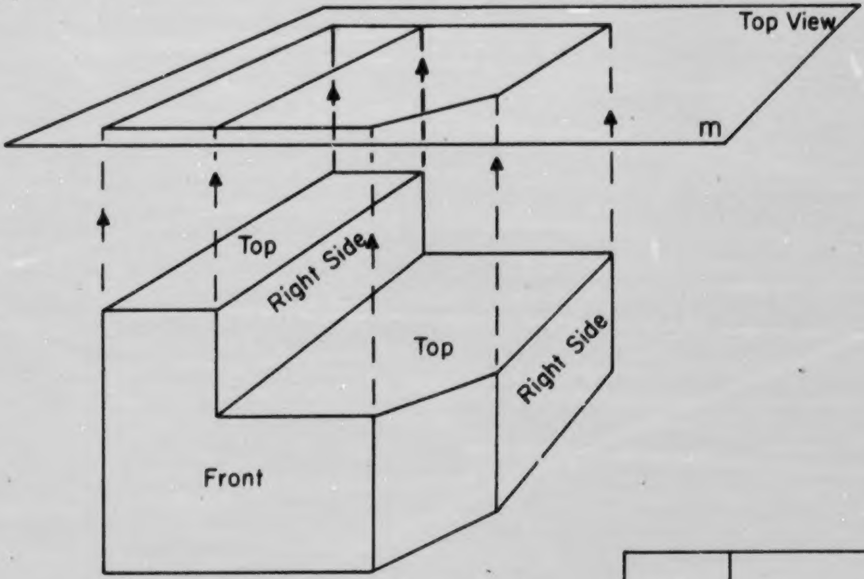
Problem Set 10-3

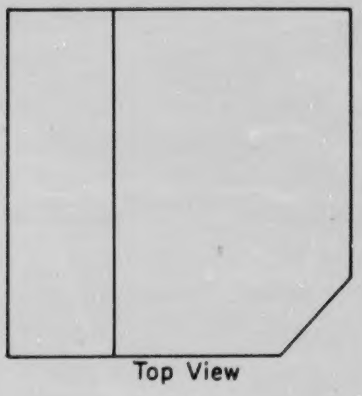
1. Using the kind of projection explained in Section 10-3 answer the following:
   a. Is the projection of a point always a point?
   b. Is the projection of a segment always a segment?
   c. Can the projection of an angle be a ray? A line? An angle?
   d. Can the projection of an acute angle be an obtuse angle?
   e. Is the projection of a right angle always a right angle?
   f. Can the length of the projection of a segment be greater than the length of the segment?
2. a. If two segments are congruent will their projections be congruent?
   b. If two lines intersect can their projections be two parallel lines?
   c. If two lines do not intersect can their projections be two intersecting lines?
   d. If two segments are parallel and congruent, will their projections be congruent?

3. Given the figure with $\overline{AB}$ not in plane m, $\overline{XY}$ the projection of $\overline{AB}$ into plane m, M the mid-point of $\overline{AB}$, and N the projection of M, prove N is the mid-point of $\overline{XY}$.



4.


Top View
m
Top
Right Side
Top
Right Side
Front

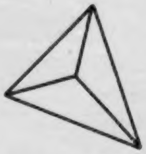


In mechanical drawing the top view or "plan" of a solid may be considered the projection of the various segments of the solid into a horizontal plane m, as shown in perspective at the left. The top view as it would actually be drawn is shown at the right. (No attempt is made here to give dimensions to the segments.)

a. Sketch a front view of the solid shown above - that is, sketch the result of projecting the segments of the solid into any plane parallel to its front face.

b. Sketch the right side view of the solid.

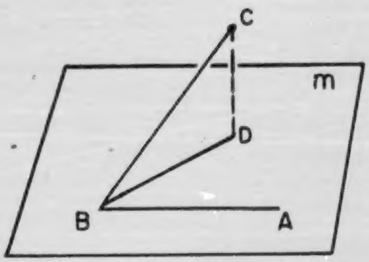
5. The projection of a tetrahedron (triangular pyramid) into the plane of its base may look like the figure at the right. How else may it appear?



6. Given: $\overline{BD}$ is the projection of $\overline{BC}$ into plane m. $\overline{AB}$ lies in plane m and $\angle ABC$ is a right angle.

Prove: $\angle ABD$ is a right angle.

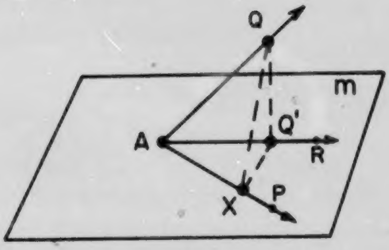
(Hint: Let $\overline{BE}$ be perpendicular to plane m.)



*7. Given: $\overrightarrow{AQ}$ has projection $\overrightarrow{AR}$ in plane m. $\overrightarrow{AP}$ is any other ray from A in plane m. (Note: $\angle QAR$ is called the angle that $\overrightarrow{AQ}$ makes with plane m.)

Prove: $m\angle QAR < m\angle QAP$.

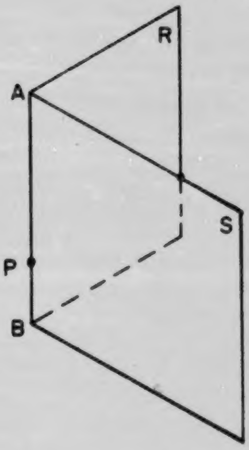
(Hint: Let Q' be the projection of Q into m. On $\overline{AP}$ choose X so that $AX = AQ'$. Draw $\overline{QQ'}$, $\overline{Q'X}$ and $\overline{QX}$.)



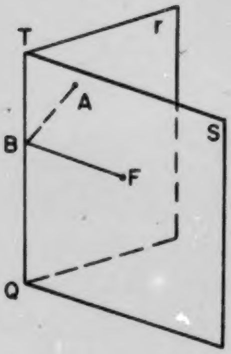
*8. If the diagonal of a cube is perpendicular to a given plane, sketch the projection into the plane of all the edges of the cube. (No proof required.)

Review Problems

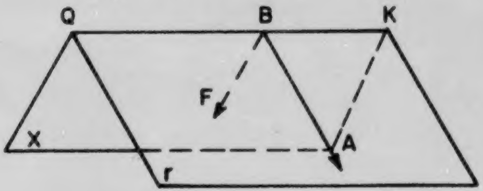
1. Suppose $\angle R\text{-}AB\text{-}S$ is an acute dihedral angle with P a point on its edge. Can rays $\overrightarrow{PX}$ and $\overrightarrow{PY}$ be chosen in the two faces so that
   a. $\angle XPY$ is acute?
   b. $\angle XPY$ is obtuse?
   c. $\angle XPY$ is right?



2. Planes r and s intersect in $\overleftrightarrow{TQ}$. B is a point between T and Q. $\overline{AB}$ is in r. $m\angle TBA = 40$. $\overline{FB}$ is in s. $m\angle FBQ = 90$. Is $\angle ABF$ a plane angle of dihedral $\angle TQ$? Can you determine $m\angle ABF$? If so, state a theorem to support your conclusion.



3. Planes x and r intersect in $\overleftrightarrow{QK}$. B is a point between K and Q. $\overrightarrow{BA}$ is in r. $\overrightarrow{BF}$ is in x. $m\angle ABK = 90$. $m\angle QBF = 90$. Is $\angle FBA$ a plane angle of dihedral $\angle QK$? If your answer is "Yes", state a theorem or definition to support your conclusion. If $m\angle ABF = 80$, is $r \perp x$? If $r \perp x$, what is $m\angle ABF$?



4. Indicate whether each of the following statements is true in all cases (A), true in some cases and false in others (S), or true in no case (N).

   a. Two lines parallel to the same plane are perpendicular to each other.

   b. If a plane intersects each of two intersecting planes, the lines of intersection are parallel.

   c. If a line lies in a plane, a perpendicular to the line is perpendicular to the plane.

d. If two planes are parallel, any line in one of them is parallel to the other.

e. If two planes are parallel to the same line they are parallel to each other.

f. Two lines perpendicular to the same line at the same point are perpendicular to each other.

g. If two intersecting planes are each perpendicular to a third plane, their line of intersection is perpendicular to the third plane.

h. The projection of a segment is a segment.

i. The projection of a right angle is a right angle.

j. Congruent segments have congruent projections.

k. Two lines are parallel if they are both perpendicular to the same line.

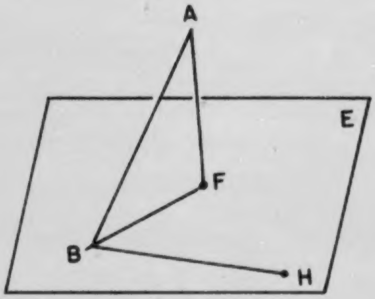
l. If a plane is perpendicular to each of two lines, the two lines are coplanar.

m. If a plane intersects two other planes in parallel lines, then the two planes are parallel.

n. If a plane intersects the faces of a dihedral angle, the intersection is called a plane angle of the dihedral angle.

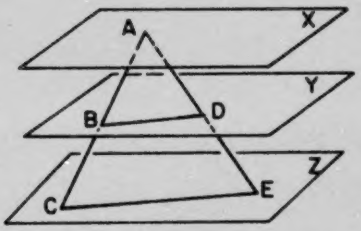
5. Given: F is the projection of point A into plane E. $\overline{BH}$ lies in plane E. $\angle$ FBH is a right angle.

Prove: $\angle$ ABH is a right angle.



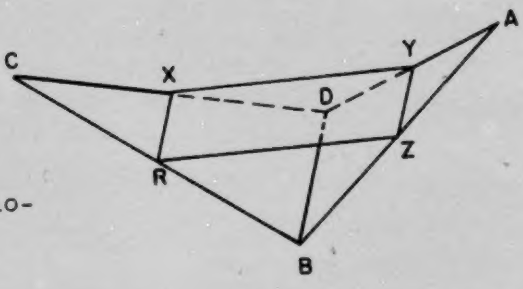
6. Given: Planes X, Y and Z are parallel as shown, with $\overline{CE}$ in Z, and A in X. $\overline{AC}$ cuts Y in B and $\overline{AE}$ cuts Y in D. AB = BC. AC = CE.

Prove: BD = BA.



7. Given: R, Z, Y, X are the mid-points of the respective sides $\overline{CB}$, $\overline{BA}$, $\overline{AD}$, $\overline{DC}$ of the non-planar quadrilateral CBAD.

Prove: RZYX is a parallelogram.

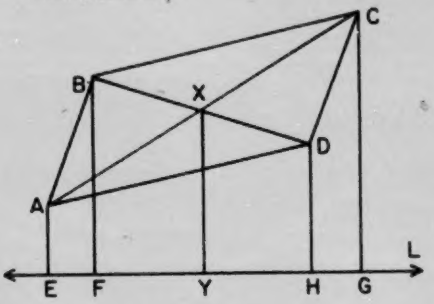


*8. In the following incomplete statement it is possible to fill in the solid blanks with "line" or "plane" and the dotted blanks with || or ⊥ in eight ways so as to make the completed statement true: Give five of these ways.

If ________A is--------to ______C, and _______B is ------to C, then A and B are--------.

*9. Given: ABCD is a parallelogram. Each of $\overline{AE}$, $\overline{BF}$, $\overline{XY}$, $\overline{DH}$ and $\overline{CG}$ are perpendicular to L, L is in the plane of parallelogram ABCD.

Prove: AE + CG = BF + DH.



Appendix I

A CONVENIENT SHORTHAND

There was a time when algebra was all written out in words. In words, you might state an algebraic problem in the following way:

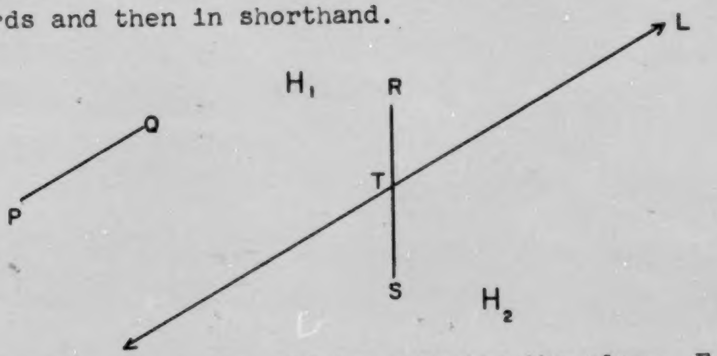
"If you square a certain number, add five times the number, and then subtract six, the result is zero. What are possibilities for this number?"

This problem can be more briefly stated in the following form:

"Find the roots of the equation $x^2 + 5x - 6 = 0$."

The notation of algebra is a very convenient shorthand. A similar shorthand has been invented for talking about sets. It saves a lot of time and space, once you get used to it, and it is all right to use it in your written work, unless your teacher objects.

Let us start with a picture, and say various things about it first in words and then in shorthand.



Here we see a line L, which separates the plane E into two half-planes H_1 and H_2. Now let us say some things in two ways.

In Words	In Shorthand
1. The segment $\overline{PQ}$ lies in H_1.	1. $\overline{PQ} \subset H_1$.
2. The intersection of $\overline{RS}$ and L is T.	2. $\overline{RS} \cap L = T$.

The shorthand expression $\overline{PQ} \subset H_1$ is pronounced in exactly the same way as the expression on the left of it. In general, when we write $A \subset B$, this means that the set A lies in the set B.

An expression of the type $A \cap B$ denotes the intersection of the sets A and B. The symbol " $\cap$ " is pronounced "cap," because it looks a little like a cap. Notice that the sets $\overline{PQ}$ and $\overline{RS}$ do not intersect. If we agree to write 0 for the empty set, then we can express this fact by writing

$$\overline{PQ} \cap \overline{RS} = 0.$$

Similarly,

$$\overline{PQ} \cap L = 0$$

and

$$\overline{PQ} \cap H_2 = 0.$$

Of course, $\overline{PQ}$ is a set which lies in H_1. But the point P above is a member of H_1. We write this in shorthand like this

$$P \in H_1.$$

This is pronounced "P belongs to H_1."

The union of two sets A and B is written as $A \cup B$. This is pronounced "A cup B." In the same way, we write $A \cup B \cup C$ for the union of three sets. For example, in the figure above, the plane E is the union of H_1, H_2 and L. We can therefore write

$$E = H_1 \cup H_2 \cup L.$$

Notice that here (as everywhere else), a formula involving the sign "=" means that the things on the left and right of "=" are the same thing. The sign "=" is simply an abbreviation of the word "is", as in the expression 2 + 2 = 4, which says that two plus two is four.

Problem Set I

Consider the sets, A, B, C, and so on, defined in the following way:

A is the set of all doctors.

B is the set of all lawyers.

C is the set of all tall people.

D is the set of all people who can play the violin.

E is the set of all people who make a lot of money.

F is the set of all basketball players.

Write shorthand expressions for the following statements:

1. All basketball players are tall.
2. No doctor is a lawyer.
3. No violinist makes a lot of money, unless he is tall.
4. No basketball player is a violinist.
5. Everyone who is both a doctor and a lawyer can also play the violin.
6. Every basketball player who can play the violin makes a lot of money.
7. The man X is a tall violinist.
8. The man Y is a prosperous lawyer.
9. The man Z is a tall basketball player.

Appendix II

POSTULATES OF ADDITION AND MULTIPLICATION

The methods of manipulating real numbers by means of the operation of addition and multiplication, and the related operations of subtraction and division, are all determined by the following eleven postulates. In the statement of these postulates and the proofs of the following theorems it is to be understood that all the letters are real numbers.

A-1. (Closure under Addition.) $x + y$ is always a real number.

A-2. (Associative Law for Addition.) $x + (y + z) = (x + y) + z$.

A-3. (Commutative Law for Addition.) $x + y = y + x$.

A-4. (Existence of 0.) There is a unique number 0 such that $x + 0 = x$ for every x.

A-5. (Existence of Negatives.) For each x there is a unique number $-x$ such that $x + (-x) = 0$.

M-1. (Closure under Multiplication.) xy is always a real number.

M-2. (Associative Law for Multiplication.) $x(yz) = (xy)z$.

M-3. (Commutative Law for Multiplication.) $xy = yx$.

M-4. (Existence of 1.) There is a unique number 1 such that $x \cdot 1 = x$ for every x.

M-5. (Existence of Reciprocals.) For each number x other than 0 there is a unique number $\frac{1}{x}$ such that $x \cdot \frac{1}{x} = 1$.

D. (Distributive Law.) $x(y + z) = xy + xz$.

The following basic theorems will illustrate how these postulates are used in simple cases.

Theorem II-1. If $b = -a$, then $-b = a$.

Proof: By A-5, $b = -a$ means the same as $a + b = 0$. By A-3 this is the same as $b + a = 0$. By A-5, this is the same as $a = -b$.

Another way of stating this theorem is that $-(-a) = a$.

Theorem II-2. For any a, $a \cdot 0 = 0$.

Proof:

$$\begin{aligned} a &= a \cdot 1 && \text{(M-4)} \\ &= a(1 + 0) && \text{(A-4)} \\ &= a \cdot 1 + a \cdot 0 && \text{(D)} \\ &= a + a \cdot 0 && \text{(M-4)} \end{aligned}$$

Hence by (A-4), $a \cdot 0 = 0$.

Theorem II-3. $a(-b) = -(ab)$.

Proof:

$$\begin{aligned} ab + a(-b) &= a[b + (-b)] && \text{(D)} \\ &= a \cdot 0 && \text{(A-5)} \\ &= 0 && \text{(Th. II-2)} \end{aligned}$$

Hence by A-4, $a(-b) = -(ab)$.

As a special case of this theorem we have $a(-1) = -a$.

Definition. $x - y$ shall mean $x + (-y)$. Note that by this definition $a - a = 0$.

Theorem II-4. If $a + b = c$, then $a = c - b$.

Proof: If $a + b = c$, then

$$(a + b) + (-b) = c + (-b)$$

$$\begin{aligned} (a + b) + (-b) &= a + [b + (-b)] && \text{(A-2)} \\ &= a + 0 . && \text{(A-5)} \\ &= a && \text{(A-4)} \end{aligned}$$

Hence $a = c + (-b) = c - b$ by definition.

Theorem II-5. If $ab = 0$, then either $a = 0$ or $b = 0$.

Proof: To prove the theorem it will be enough to show that if $a \neq 0$ then $b = 0$. So suppose $a \neq 0$. Then $\frac{1}{a}$ exists, by M-5. Therefore,

$$\frac{1}{a}(ab) = \frac{1}{a} \cdot 0 = 0 \qquad \text{(Th. A-II-2)}$$

also,

$$\begin{aligned} \frac{1}{a}(ab) &= (\frac{1}{a} \cdot a)\, b && \text{(M-2)} \\ &= 1 \cdot b && \text{(M-5)} \\ &= b \cdot 1 && \text{(M-3)} \\ &= b && \text{(M-4)} \end{aligned}$$

Therefore $b = 0$.

Theorem II-6. (Cancellation Law.) If $ab = ac$ and $a \neq 0$ then $b = c$.

Proof: If $ab = ac$ then $ab - ac = 0$. By Theorem A-II-3 this is the same as $ab + a(-c) = 0$, or, by D, as $a(b - c) = 0$. Since $a \neq 0$ we get, by applying Theorem II-5, that $b - c = 0$. Hence $b = c$.

These are just a few examples of the use of the postulates in proving basic algebraic theorems. Ordinarily we don't use the postulates directly but make use of such properties as those stated in Theorems II-4 and II-6 in our algebraic work.

Problem Set II

1. Prove each of the following theorems.
   a. $(-a)(-b) = ab$.
   b. $a(b-c) = ab - ac$.
   c. If $a - b = c$, then $a = b + c$.
   d. $(a + b)(c + d) = ac + ad + bc + bd$. (Hint: As a first step apply D, regarding $(a + b)$ as a single number.)
2. Given the definitions:
   $x^2 = x \cdot x$,
   $2 = 1 + 1$,
   prove that
   $$(a + b)^2 = a^2 + 2ab + b^2.$$
3. Prove: $(a + b)(a-b) = a^2 - b^2$.
4. Definition: $\frac{a}{b} = ab^{-1}$.
   Prove each of the following:
   a. $(ab)^{-1} = a^{-1}b^{-1}$.
   b. $\frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd}$.
   c. $\frac{a}{b} = \frac{ac}{bc}$.
   d. $(-a)^{-1} = -(a^{-1})$.
   e. $\frac{a}{-b} = \frac{-a}{b} = -\frac{a}{b}$.
   f. $\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$
   g. $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$.

Appendix III

RATIONAL AND IRRATIONAL NUMBERS

III-1. How to Show That a Number is Rational.

By definition a number is rational if it is the ratio of two integers. Therefore, it we want to prove that a number x is rational, we have to produce two integers p and q, such that $\frac{p}{q} = x$. Here are some examples:

(1) The number $x = \frac{1}{2} + \frac{3}{7}$ is rational, because

$$\frac{1}{2} + \frac{3}{7} = \frac{7 + 6}{14} = \frac{13}{14} .$$

Therefore $x = \frac{p}{q}$, where $p = 13$ and $q = 14$.

(2) The number $x = 1.23$ is rational, because

$$1.23 = \frac{123}{100},$$

which is the ratio of the two integers 123 and 100.

(3) If the number x is rational, then so is the number $2x$. (That is, twice a rational number is always rational.) For if

$$x = \frac{p}{q},$$

where p and q are integers, then

$$2x = \frac{2p}{q} ,$$

where the numerator $2p$ and the denominator q are both integers.

(4) If the number x is rational, then so is the number $x + \frac{2}{3}$. For if

$$x = \frac{p}{q} ,$$

then

$$x + \frac{2}{3} = \frac{p}{q} + \frac{2}{3} = \frac{3p + 2q}{3q}.$$

where the numerator and denominator are both integers.

(5) If x is a rational number, then so is $x^2 + x$. For if

$$x = \frac{p}{q} ,$$

then

$$x^2 + x = \frac{p^2}{q^2} + \frac{p}{q} = \frac{p^2 + pq}{q^2}$$

where the numerator and denominator are integers.

Problem Set III-1

1. Show that .2351 is a rational number.
2. Show that $\frac{2}{3} + \frac{5}{7}$ is rational.
3. Show that if x is a rational number, then so is $x - 5$.
4. Show that if x is rational, then so is $2x - 7$.
5. Show that $\frac{1}{3} + \frac{1}{17}$ is rational.
6. Show that the sum of any two rational numbers is a rational number.
7. Show that $(\frac{17}{19})(\frac{23}{47})$ is rational.
8. Show that the product of any two rational numbers is a rational number.
9. Show that $\frac{3}{17} + \frac{23}{7}$ is rational.
10. Show that the quotient of any two rational numbers is a rational number, as long as the divisor is not zero.
11. Given that $\sqrt{2}$ is irrational, show that $\frac{\sqrt{2}}{2}$ is also irrational. (Hint: This problem is a lot easier, now that you understand about indirect proofs.)
12. Given that π is irrational, show that $\frac{\pi}{5}$ is also irrational.
13. Show that the reciprocal of every rational number different from zero is rational.
14. Show that the reciprocal of every irrational number different from zero is irrational.
15. Is it true that the sum of a rational number and an irrational number is always irrational? Why or why not?
16. Is it true that the sum of two irrational numbers is always irrational? Why or why not?
17. How about the product of a rational number and an irrational number?

III-2. Some Examples of Irrational Numbers.

In the previous section, we proved that under certain conditions a number must be rational. In some of the problems, you showed that *starting with an irrational number* we could get more irrational numbers in various ways. In all this we left one very important question unsettled: are there any irrational numbers? We shall settle this question by showing that a particular number, namely $\sqrt{2}$, cannot be expressed as the ratio of any two integers.

To prove this, we first need to establish some of the facts about squares of odd and even integers. Every integer is either even or odd. If n is even, then n is twice some integer k, and we can write

$$n = 2k.$$

If n is odd, then when we divide by 2 we get a quotient k and a remainder 1, so that

$$\frac{n}{2} = k + \frac{1}{2}.$$

Therefore, we can write

$$n = 2k + 1.$$

These are the typical formulas for even numbers and odd numbers respectively. For example,

$$\begin{array}{ll} 6 = 2 \cdot 3 & n = 6,\ k = 3 \\ 7 = 2 \cdot 3 + 1 & n = 7,\ k = 3 \\ 8 = 2 \cdot 4 & n = 8,\ k = 4 \\ 9 = 2 \cdot 4 + 1 & n = 9,\ k = 4, \end{array}$$

and so on. The following theorem is easy to prove:

Theorem III-1. The square of every odd number is odd.

Proof: If n is odd, then we can write

$$n = 2k + 1,$$

where k is an integer. Squaring both sides, we get

$$\begin{aligned} n^2 &= (2k)^2 + 2 \cdot 2k + 1 \\ &= 4k^2 + 4k + 1. \end{aligned}$$

The right-hand side must be odd, because it is written in the form

$$2 \cdot [2k^2 + 2k] + 1;$$

that is, it is twice an integer, plus 1. Therefore, n^2 is odd, which was to be proved.

From Theorem III-1 we can quickly get another theorem:

Theorem III-2. If n^2 is even, then n is even.

Proof: If n were odd, then n^2 would be odd, which is false. Therefore n is even.

Notice that this is an indirect proof.

We are now ready to begin the proof of

Theorem III-3. $\sqrt{2}$ is irrational.

Proof: The proof will be indirect. We begin by making the assumption that $\sqrt{2}$ is rational. We will show that this leads to a contradiction.

Step 1. Supposing that $\sqrt{2}$ is rational, it follows that 2 can be expressed as

$$\sqrt{2} = \frac{p}{q}$$

where the fraction $\frac{p}{q}$ is in lowest terms.

The reason is that if $\sqrt{2}$ can be expressed as a fraction at all, then we can reduce the fraction to lowest terms by dividing out any common factors of the numerator and denominator.

We therefore have

$$\sqrt{2} = \frac{p}{q} ,$$

in lowest terms. This gives

$$2 = \frac{p^2}{q^2} ,$$

which in turn gives

$$p^2 = 2q^2 .$$

Step 2. p^2 is even.

Because p^2 is twice an integer.

Step 3. p is even.

By Theorem III-2.

We therefore set $p = 2k$. Substituting in the formula at the end of Step 1, we get

$$(2k)^2 = 2q^2 ,$$

which means that

$$4k^2 = 2q^2 .$$

Therefore

$$q^2 = 2k^2 .$$

Step 4. q^2 is even.

Because q^2 is twice an integer.

Step 5. q is even.

By Theorem III-2.

We started by assuming that $\sqrt{2}$ was rational. From this we got $\sqrt{2} = \frac{p}{q}$, in lowest terms. From this we have proved that p and q were both even. Therefore $\frac{p}{q}$ was not in lowest terms, after all. This contradiction shows that our initial assumption must have been wrong, that is, $\sqrt{2}$ must not be rational.

Problem Set III-2

These problems are harder than most of the problems in the text.

1. Adapt the proof that $\sqrt{2}$ is irrational, so as to get a proof that $\sqrt{3}$ is irrational. (Hint: Start with the fact that every integer has one of the forms

$$n = 3k$$
$$n = 3k + 1$$
$$n = 3k + 2,$$

and then prove a theorem corresponding to Theorem III-2.)

2. Obviously nobody can prove that $\sqrt{4}$ is irrational, because $\sqrt{4} = 2$. If you try to "prove" this by adapting the proof for $\sqrt{2}$, at what point does the "proof" break down?

3. Show that $\sqrt[3]{2}$ is irrational.

Actually, the square root of an integer is either another integer or an irrational number; that is, $\sqrt{n}$ either "comes out very even" or "comes out very uneven." The proof of this fact, however, requires more mathematical technique than we now have at our disposal. Problems like this are solved in a branch of mathematics called the Theory of Numbers.

Appendix IV
SQUARES AND SQUARE ROOTS

Everybody knows what it means to square a number: you multiply the number by itself. The facts about square roots, however, are considerably trickier, and the language in which most people talk about them is very confusing. Here we will try to state the facts and point out the pit-falls.

To say that x is a square root of a means that

$$x^2 = a.$$

For example,

2 is a square root of 4,
3 is a square root of 9,
-2 is a square root of 4,
-3 is a square root of 9,

and so on. You may wonder why we did not abbreviate these statements by using radical signs. The reason (as we shall soon see) is that radical signs mean something slightly different.

The following is a fundamental fact about the real number system:

> Every positive number has exactly one positive square root.

For example, $2^2 = 4$, and no other positive number is a root of the equation $x^2 = 4$. $4^2 = 16$, and no other positive number is a root of the equation $x^2 = 16$. And so on.

Of course, if x is a square root of a, then so is $-x$, because $(-x)^2 = x^2$. Therefore every positive number has exactly two square roots, one positive and the other negative. The meaning of the radical sign is defined this way:

If a is positive, then $\sqrt{a}$ denotes the positive square root of a.

We provide further that $\sqrt{0} = 0$.
For example,

$$\sqrt{4} = 2,$$
$$\sqrt{9} = 3,$$
$$\sqrt{16} = 4,$$

and so on. To indicate the other square root -- that is, the negative one -- we simply put a minus sign in front of the radical sign. For example:

4 has two square roots, 2 and -2.
3 has two square roots, $\sqrt{3}$ and $-\sqrt{3}$.
7 has two square roots, $\sqrt{7}$ and $-\sqrt{7}$.

The following two statements look alike, but in fact they are different:

(1) x is a square root of a.
(2) $x = \sqrt{a}$.

The first statement means merely that $x^2 = a$. The second statement means not only that $x^2 = a$, but also that $x \geq 0$. Therefore the second statement is not simply a short-hand form of the first.

Let us now investigate the expression $\sqrt{x^2}$, where x is not equal to zero. There are two possibilities:

I. If $x > 0$, then x is the positive square root of x^2, and we can write

$$\sqrt{x^2} = x.$$

II. If $x < 0$, then x is the negative square root of x^2, and it is $-x$ that is the positive square root of x^2. Therefore, for $x < 0$, we have

$$\sqrt{x^2} = -x.$$

The equation $\sqrt{x^2} = x$ looks so appealing that it seems almost like a law of nature. In fact, however, this equation holds true only half of the time: it is always true when $x \geq 0$, and it is never true when $x < 0$.

Fitting together cases I and II, we see that for every x without exception we have

$$\sqrt{x^2} = |x|.$$

To see this, you should check it against the definition of $|x|$, in Section 2-3.

Problem Set IV

Which of the following statements are true? Why or why not?

1. $\sqrt{9} = 3.$
2. $\sqrt{9} = -3.$
3. $\sqrt{2} = \pm 1.414.$ (Approximately.)
4. $\sqrt{2} = 1.414.$ (Approximately.)
5. $\sqrt{25} = \pm 5.$
6. $\sqrt{25} = 5.$

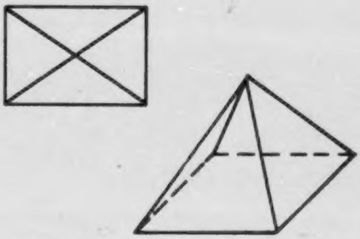
For what values of the unknowns (if any) do the following equations hold true? Why?

7. $\sqrt{(x-1)^2} = x - 1.$
8. $\sqrt{(x-1)^2} = 1 - x.$
9. $\sqrt{(x-1)^2} = |x - 1|.$
10. $\sqrt{(x-1)^2} = -|x - 1|.$
11. $\sqrt{(x+3)^4} = (x + 3)^2.$
12. $\sqrt{(x+3)^4} = -(x + 3)^2.$
13. $\sqrt{(x+3)^4} = |(x + 3)^2|.$
14. $\sqrt{(x+3)^4} = -|(x + 3)^2|.$

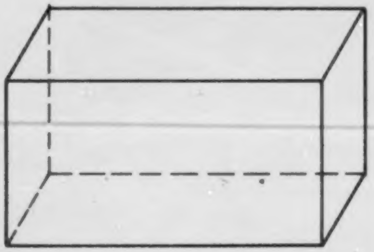
Appendix V
HOW TO DRAW FIGURES IN 3-SPACE

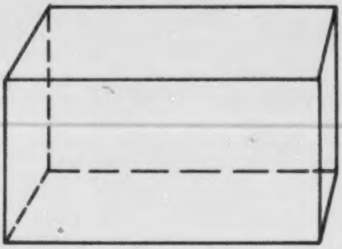
V-1. Simple Drawing.

A course in mechanical drawing is concerned with precise representation of physical objects seen from different positions in space. In geometry we are concerned with drawing only to the extent that we use sketches to help us do mathematical thinking. There is no one correct way to draw pictures in geometry, but there are some techniques helpful enough to be in rather general use. Here, for example, is a technically correct drawing of an ordinary pyramid, for a person can argue that he is looking at the pyramid from directly above. But careful ruler drawing is not as helpful as this very crude free-hand sketch. The first drawing does not suggest 3-space; the second one does.



The first part of this discussion offers suggestions for simple ways to draw 3-space figures. The second part introduces the more elaborate technique of drawing from perspective. The difference between the two approaches is suggested by these two drawings of a rectangular box.



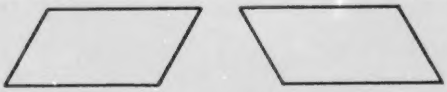


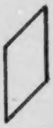
In the first drawing the base is shown by an easy-to-draw parallelogram. In the second drawing, the front base edge and the back base edge are parallel, but the back base edge is drawn shorter under the belief that the shorter length will suggest "more remote".

No matter how a rectangular box is drawn, some sacrifices must be made. All angles of a rectangular solid are right angles, but in each of the drawings shown above two-thirds of the angles do not come close to indicating ninety degrees when measured with a protractor. We are willing to give up the drawing of right angles that look like right angles in order that we make the figure as a whole more suggestive.

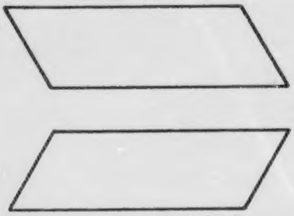
You already know that a plane is generally pictured by a parallelogram.

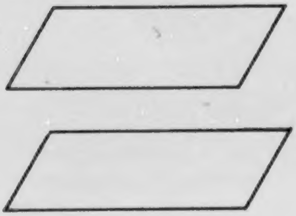
It seems reasonable to draw a horizontal plane in either of the ways shown, and to draw a vertical plane like this.



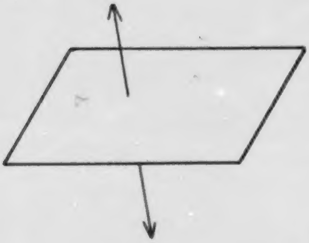


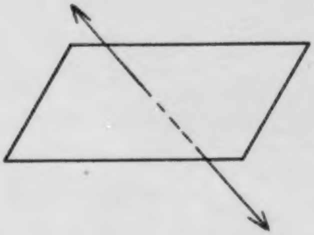
If we want to indicate two parallel planes, however, we can not be effective if we just draw any two "horizontal" planes. Notice how the drawing to the right below improves upon the one to the left. Perhaps you prefer still another kind of drawing.



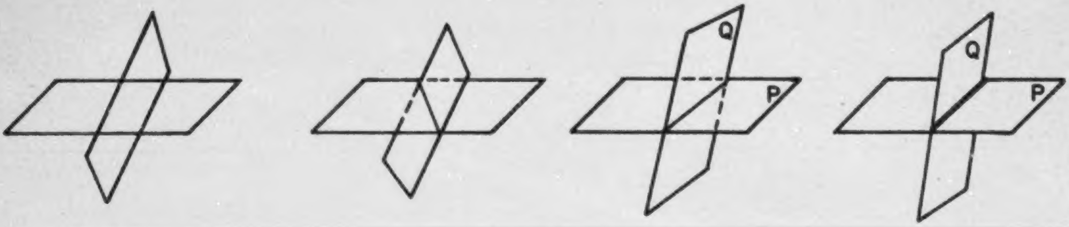


Various devices are used to indicate that one part of a figure passes behind another part. Sometimes a hidden part is simply omitted, sometimes it is indicated by dotted lines. Thus a line piercing a plane may be drawn in either of the two ways:



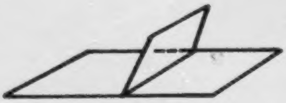


Two intersecting planes are illustrated by each of these drawings.

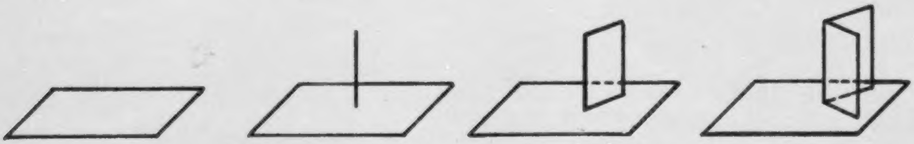


The second is better than the first because the line of intersection is shown and parts concealed from view are dotted. The third and fourth drawings are better yet because the line of intersection is visually tied in with plane P as well as plane Q by the use of parallel lines in the drawing.

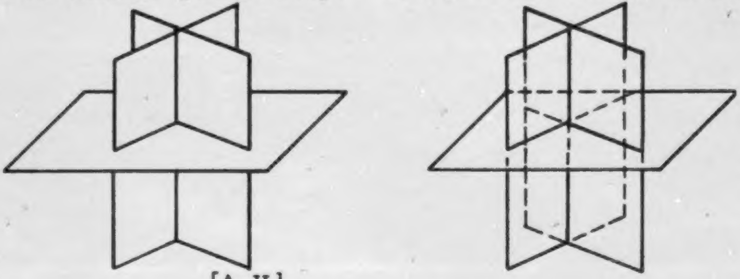
Here is a drawing which has the advantage of simplicity and the disadvantage of suggesting one plane and one half-plane. In any case a line of intersection is a particularly important part of a figure.



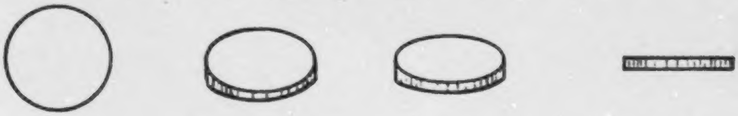
Suppose that we wish to draw two intersecting planes each perpendicular to a third plane. An effective procedure is shown by this step-by-step development.



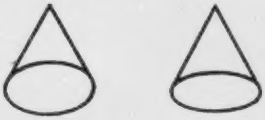
Notice how the last two planes drawn are built on the line of intersection. A complete drawing showing all the hidden lines is just too involved to handle pleasantly. The picture below is much more suggestive.



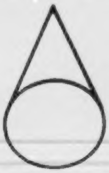
A dime, from different angles, looks like this:



Neither the first nor the last is a good picture of a circle in 3-space. Either of the others is satisfactory. The thinner oval is perhaps better to use to represent the base of a cone.



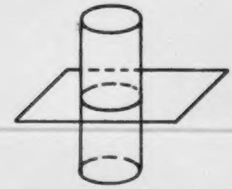
Certainly nobody should expect us to interpret the figure shown below as a cone.



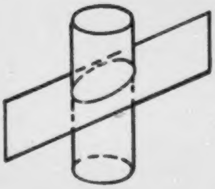
A few additional drawings, with verbal descriptions, are shown.

A line parallel to a plane.

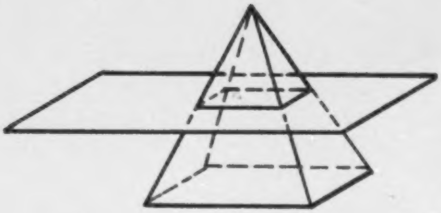
A cylinder cut by a plane parallel to the base.



A cylinder cut by a plane not parallel to the base.



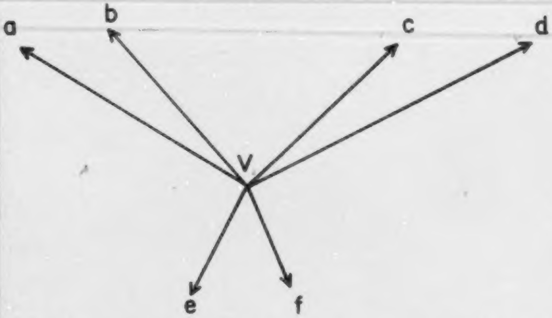
A pyramid cut by a plane parallel to the base.

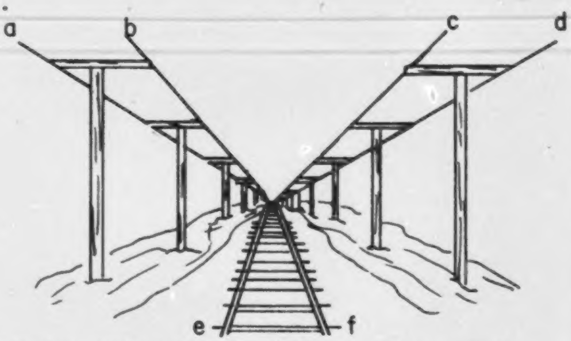


It is important to remember that a drawing is not an end in itself but simply an aid to our understanding of the geometrical situation. We should choose the kind of picture that will serve us best for this purpose, and one person's choice may be different from another.

V-2. Perspective.

The rays a, b, c, d, e, f in the left-hand figure below suggest coplanar lines intersecting at V; the corresponding rays in the right-hand figure suggest parallel lines in a three-dimensional drawing. Think of a railroad track and telephone poles as you look at the right-hand figure.





The right-hand figure suggests certain principles which are useful in making perspective drawings.

1) A set of parallel lines which recede from the viewer are drawn as concurrent rays; for example, rays a, b, c, d, e, f. The point, on the drawing, where the rays meet is known as the "vanishing point".

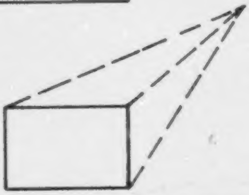
2) Congruent segments are drawn smaller when they are farther from the viewer. (Find examples in the drawing.)

3) Parallel lines which are perpendicular to the line of sight of the viewer are shown as parallel lines in the drawing. (Find examples in the drawing.)

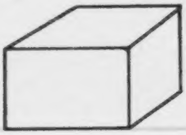
A person does not need much artistic ability to make use of these three principles.

The steps to follow in sketching a rectangular solid are shown below.

Draw the front face as a rectangle.

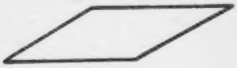


Select a vanishing point and draw segments from it to the vertices. Omit segments that cannot be seen.

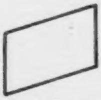


Draw edges parallel to those of the front face. Finally erase lines of perspective.

Under this technique a single horizontal plane can be drawn as the top face of the solid shown above.



A single vertical plane can be represented by the front face or the right-hand face of the solid.



After this brief account of two approaches to the drawing of figures in 3-space we should once again recognize the fact that there is no one correct way to picture geometric ideas. However, the more "real" we want our picture to appear, the more attention we should pay to perspective. Such an artist as Leonardo da Vinci paid great attention to perspective. Most of us find this done for us when we use ordinary cameras.

See some books on drawing or look up "perspective" in an encyclopedia if you are interested in a detailed treatment.

Appendix VI

PROOFS OF THEOREMS ON PERPENDICULARITY

In Section 8-3 two theorems are stated, which, between them, cover all cases of existence and uniqueness involved in the perpendicularity of a line and a plane. As stated there, eight separate items must be proved to establish the proofs of these two theorems. Here we will state these items and prove those which have not already been proved.

We first restate the two theorems.

Theorem 8-9. Through a given point there is one and only one plane perpendicular to a given line.

Theorem 8-10. Through a given point there is one and only one line perpendicular to a given plane.

We now consider the eight proofs, in a systematic order. Read the statements carefully, for there are only slight differences in their wording: the presence or absence of a "not", the substitution of "most" for "least", or the interchange of "line" and "plane".

Theorem VI-1. Through a given point on a given line there is at least one plane perpendicular to the line.

This is Theorem 8-4, which is proved in the text.

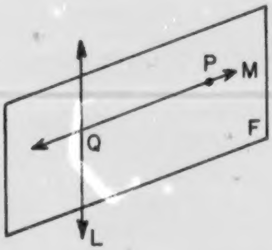
Theorem VI-2. Through a given point on a given line there is at most one plane perpendicular to the line.

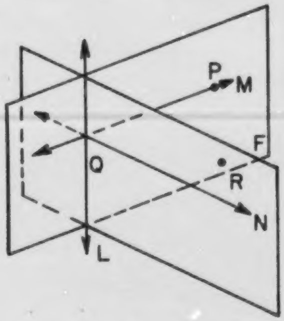
This is Theorem 8-6, which is proved in the text.

Theorem VI-3. Through a given point not on a given line there is at least one plane perpendicular to the given line.

Given: Line L and point P not on L.

To prove: There is a plane E through P, with $E \perp L$.





Proof:

(1) There is a line M through P perpendicular to L (Theorem 6-4). Let M and L intersect at Q, and lie in the plane F (Theorem 3-4).

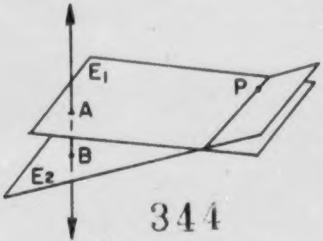
(2) There is a point R (Figure 2) not in F (Postulate 5b). Let G be the plane containing L and R (Theorem 3-3).

(3) In G there is a line N perpendicular to L at Q (Theorem 6-1).

(4) Let E be the plane containing M and N. Then $E \perp L$ by Theorem 8-3.

Theorem VI-4. Through a given point not on a given line there is at most one plane perpendicular to the given line.

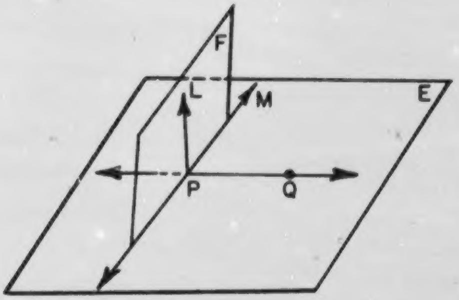
Proof: Suppose that there are two planes E_1 and E_2, each perpendicular to line L and each containing point P. If E_1 and E_2 intersect L in the same point Q, we have two planes perpendicular to L at Q, and this contradicts Theorem VI-2.



On the other hand, if E_1 and E_2 intersect L in distinct points A and B, then $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ are distinct lines through P perpendicular to L, contradicting Theorem 6-4. Either way, we get a contradiction, and so we cannot have two planes through P perpendicular to L.

This finishes the proof of Theorem 8-9. The next four theorems, which read like the previous four with "line" and "plane" interchanged, will prove Theorem 8-10.

Theorem VI-5. Through a given point in a given plane there is at least one line perpendicular to the plane.



Proof: Let P be a point in plane E. By Postulate 5a there is another point Q in E. Let plane F be perpendicular to $\overleftrightarrow{PQ}$ at P (Theorem VI-1).

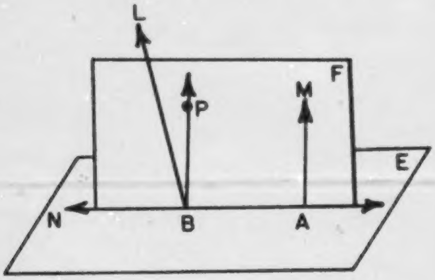
Since F intersects E (at P) their intersection is a line M, by Postulate 8. Let L be a line in F, perpendicular to M (Theorem 6-1).

Since $F \perp \overleftrightarrow{PQ}$, and L lies in F and contains P, we have, from the definition of a line perpendicular to a plane, that $L \perp \overleftrightarrow{PQ}$. Also, from above, $L \perp M$. Hence $L \perp E$, by Theorem 8-4.

Theorem VI-6. Through a given point in a given plane there is at most one line perpendicular to the given plane.

Proof: Suppose L_1 and L_2 are distinct lines, each perpendicular to plane E at point P. L_1 and L_2 determine a plane F (Theorem 3-4) which intersects E in a line L. In F, we then have two perpendiculars to L at the same point P, contradicting Theorem 6-1.

Theorem VI-7. Through a given point not in a given plane there is at least one line perpendicular to the given plane.



Proof: Let P be a point not in plane E. Let A be any point of E, and M a line through A perpendicular to E (Theorem VI-5).

If M contains P it is the desired perpendicular.

If M does not contain P let F be the plane containing M and P (Theorem 3-3), and N the line of intersection of F and E. In F let B be the foot of a perpendicular from P to N (Theorem 6-4).

Let line L be perpendicular to E at B (Theorem VI-5). By Theorem 8-8, L and M are coplanar, and hence, L lies in F since M and B determine F.

In F, $L \perp N$, since $L \perp E$ and N lies in E. Since by Theorem 6-1 there is only one line in F perpendicular to N at B, L and $\overleftrightarrow{BP}$ must coincide. That is, L contains P and so is the desired perpendicular.

Theorem VI-8. Through a given point not in a given plane there is at most one line perpendicular to the given plane.

The proof is word for word the same as that of Theorem VI-6, except for the replacement of "at point P" by "from point P" and of "Theorem 6-1" by "Theorem 6-3".

The Meaning and Use of Symbols

General.

$=$. $A = B$ can be read as "A equals B", "A is equal to B", "A equal B" (as in "Let $A = B$"), and possibly other ways to fit the structure of the sentence in which the symbol appears. However, we should not use the symbol, $=$, in such forms as "A and B are $=$"; its proper use is *between two expressions*. If two expressions are connected by "$=$", it is to be understood that these two expressions stand for the same mathematical entity, in our case either a real number or a point set.

$\neq$. "Not equal to". $A \neq B$ means that A and B do *not* represent the same entity. The same variations and cautions apply to the use of $\neq$ as to the use of $=$.

Algebraic.

$+$, $\cdot$, $-$, $\div$. These familiar algebraic symbols for operating with real numbers need no comment. The basic postulates about them are presented in Appendix II.

$<$, $>$, $\leq$, $\geq$. Like $=$, these can be read in various ways in sentences, and $A < B$ may stand for the underlined part of "If *A is less than B*", "Let *A be less than B*", "*A less than B* implies ", etc. Similarly for the other three symbols, read "greater than", "less than or equal to", "greater than or equal to". These inequalities apply only to real numbers. Their properties are mentioned briefly in Section 2-2, and in more detail in Section 7-2.

$\sqrt{A}$, $|A|$. "Square root of A" and "absolute value of A". Discussed in Sections 2-2 and 2-3 and Appendix IV.

Geometric.

Point Sets. A single letter may stand for any suitably described point set. Thus we may speak of a point P, a line m, a half-plane H, a circle C, an angle x, a segment b, etc.

$\overleftrightarrow{AB}$. The line containing the two points A and B (P. 30).

$\overline{AB}$. The segment having A and B as end-points (P. 45).

$\overrightarrow{AB}$. The ray with A as its end-point and containing point B (P. 45).

$\angle ABC$. The angle having B as vertex and $\overrightarrow{BA}$ and $\overrightarrow{BC}$ as sides (P. 71).

$\triangle ABC$. The triangle having A, B and C as vertices (P. 72).

$\angle A\text{-}BC\text{-}D$. The dihedral angle having $\overleftrightarrow{BC}$ as edge and with sides containing A and D (P. 299).

Real Numbers.

AB. The positive number which is the distance between the two points A and B, and also the length of the segment $\overline{AB}$ (P. 34).

$m\angle ABC$. The real number between 0 and 180 which is the degree measure of $\angle ABC$ (P. 80).

Area R. The positive number which is the area of the polygonal region R (P. 320).

Relations.

$\cong$. Congruence. $A \cong B$ is read "A is congruent to B", but with the same possible variations and restrictions as $A = B$. In the text A and B may be two (not necessarily different) segments (P. 109), angles (P. 109), or triangles (P. 111).

$\perp$ Perpendicular. $A \perp B$ is read "A is perpendicular to B", with the same comment as for $\cong$. A and B may be either two lines (P. 86), two planes (P. 301), or a line and a plane (P. 219).

$\parallel$. Parallel. $A \parallel B$ is read "A is parallel to B", with the same comment as for $\cong$. A and B may be either two lines (P. 241), two planes (P. 291) or a line and a plane (P. 291).

List of Postulates

Postulate 1. (P. 30) Given any two different points, there is exactly one line which contains both of them.

Postulate 2. (P. 34) (The Distance Postulate.) To every pair of different points there corresponds a unique positive number.

Postulate 3. (P. 36) (The Ruler Postulate.) The points of a line can be placed in correspondence with the real numbers in such a way that

(1) To every point of the line there corresponds exactly one real number,

(2) To every real number there corresponds exactly one point of the line, and

(3) The distance between two points is the absolute value of the difference of the corresponding numbers.

Postulate 4. (P. 40) (The Ruler Placement Postulate.) Given two points P and Q of a line, the coordinate system can be chosen in such a way that the coordinate of P is zero and the coordinate of Q is positive.

Postulate 5. (P. 54) (a) Every plane contains at least three non-collinear points.

(b) Space contains at least four non-coplanar points.

Postulate 6. (P. 56) If two points lie in a plane, then the line containing these points lies in the same place.

Postulate 7. (P. 57) Any three points lie in at least one plane, and any three non-collinear points lie in exactly one plane. More briefly, any three points are coplanar, and any three non-collinear points determine a plane.

Postulate 8. (P. 58) If two different planes intersect, then their intersection is a line.

Postulate 9. (P. 64) (The Plane Separation Postulate.) Given a line and a plane containing it, the points of the plane that do not lie on the line form two sets such that

(1) each of the sets is convex and

(2) if P is in one set and Q is in the other then the segment $\overline{PQ}$ intersects the line.

Postulate 10. (P. 66) (The Space Separation Postulate.) The points of space that do not lie in a given plane form two sets such that

(1) each of the sets is convex and

(2) if P is in one set and Q is in the other, then the segment $\overline{PQ}$ intersects the plane.

Postulate 11. (P. 80) (The Angle Measurement Postulate.) To every angle $\angle BAC$ there corresponds a real number between 0 and 180.

Postulate 12. (P. 81) (The Angle Construction Postulate.) Let $\overrightarrow{AB}$ be a ray on the edge of the half-plane H. For every number r between 0 and 180 there is exactly one ray $\overrightarrow{AP}$, with P in H, such that $m\angle PAB = r$.

Postulate 13. (P. 81) (The Angle Addition Postulate.) If D is a point in the interior of $\angle BAC$, then $m\angle BAC = m\angle BAD + m\angle DAC$.

Postulate 14. (P. 82) (The Supplement Postulate.) If two angles form a linear pair, then they are supplementary.

Postulate 15. (P. 115) (The S.A.S. Postulate.) Given a correspondence between two triangles (or between a triangle and itself). If two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

Postulate 16. (P. 252) (The Parallel Postulate.) Through a given external point there is at most one line parallel to a given line..

Postulate 17. (P. 320) To every polygonal region there corresponds a unique positive number.

Postulate 18. (P. 320) If two triangles are congruent, then the triangular regions have the same area.

Postulate 19. (P. 320) Suppose that the region R is the union of two regions R_1 and R_2. Suppose that R_1 and R_2 intersect at most in a finite number of segments and points. Then the area of R is the sum of the areas of R_1 and R_2.

Postulate 20. (P. 322) The area of a rectangle is the product of the length of its base and the length of its altitude.

f

Postulate 21. (P. 546) The volume of a rectangular parallelepiped is the product of the altitude and the area of the base.

Postulate 22. (P. 548) (Cavalieri's Principle.) Given two solids and a plane. If for every plane which intersects the solids and is parallel to the given plane the two intersections have equal areas, then the two solids have the same volume.

List of Theorems and Corollaries

Theorem 2-1. (P. 42) Let A, B, C be three points of a line, with coordinates x, y, z. If $x < y < z$, then B is between A and C.

Theorem 2-2. (P. 43) Of any three different points on the same line, one is between the other two.

Theorem 2-3. (P. 44) Of three different points on the same line, only one is between the other two.

Theorem 2-4. (P. 46) (The Point Plotting Theorem.) Let $\overrightarrow{AB}$ be a ray, and let x be a positive number. Then there is exactly one point P of $\overrightarrow{AB}$ such that $AP = x$.

Theorem 2-5. (P. 47) Every segment has exactly one midpoint.

Theorem 3-1. (P. 55) Two different lines intersect in at most one point.

Theorem 3-2. (P. 56) If a line intersects a plane not containing it, then the intersection is a single point.

Theorem 3-3. (P. 57) Given a line and a point not on the line, there is exactly one plane containing both of them.

Theorem 3-4. (P. 58) Given two intersecting lines, there is exactly one plane containing them.

Theorem 4-1. (P. 87) If two angles are complementary, then both of them are acute.

Theorem 4-2. (P. 87) Every angle is congruent to itself.

Theorem 4-3. (P. 87) Any two right angles are congruent.

Theorem 4-4. (P. 87) If two angles are both congruent and supplementary, then each of them is a right angle.

Theorem 4-5. (P. 87) Supplements of congruent angles are congruent.

Theorem 4-6. (P. 88) Complements of congruent angles are congruent.

Theorem 4-7. (P. 88) Vertical angles are congruent.

Theorem 4-8. (P. 89) If two intersecting lines form one right angle, then they form four right angles.

Theorem 5-1. (P. 109) Every segment is congruent to itself.

Theorem 5-2. (P. 127) If two sides of a triangle are congruent, then the angles opposite these sides are congruent.

Corollary 5-2-1. (P. 128) Every equilateral triangle is equiangular.

Theorem 5-3. (P. 129) Every angle has exactly one bisector.

Theorem 5-4. (P. 132) (The A.S.A. Theorem.) Given a correspondence between two triangles (or between a triangle and itself). If two angles and the included side of the first triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

Theorem 5-5. (P. 133) If two angles of a triangle are congruent, the sides opposite these angles are congruent.

Corollary 5-5-1. (P. 133) An equiangular triangle is equilateral.

Theorem 5-6. (P. 137) (The S.S.S. Theorem.) Given a correspondence between two triangles (or between a triangle and itself.) If all three pairs of corresponding sides are congruent, then the correspondence is a congruence.

Theorem 6-1. (P. 167) In a given plane, through a given point of a given line of the plane, there passes one and only one line perpendicular to the given line.

Theorem 6-2. (P. 169) The perpendicular bisector of a segment, in a plane, is the set of all points of the plane that are equidistant from the end-points of the segment.

Theorem 6-3. (P. 171) Through a given external point there is at most one line perpendicular to a given line.

Corollary 6-3-1. (P. 172) At most one angle of a triangle can be a right angle.

Theorem 6-4. (P. 172) Through a given external point there is at least one line perpendicular to a given line.

Theorem 6-5. (P. 183) If M is between A and C on a line L, then M and A are on the same side of any other line that contains C.

Theorem 6-6. (P. 183) If M is between A and C, and B is any point not on line $\overleftrightarrow{AC}$, then M is in the interior of $\angle$ ABC.

Theorem 7-1. (P. 193) (The Exterior Angle Theorem.) An exterior angle of a triangle is larger than either remote interior angle.

Corollary 7-1-1. (P. 196) If a triangle has a right angle, then the other two angles are acute.

Theorem 7-2. (P. 197) (The S.A.A. Theorem.) Given a correspondence between two triangles. If two angles and a side opposite one of them in one triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

Theorem 7-3. (P. 198) (The Hypotenuse - Leg Theorem.) Given a correspondence between two right triangles. If the hypotenuse and one leg of one triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

Theorem 7-4. (P. 200) If two sides of a triangle are not congruent, then the angles opposite these two sides are not congruent, and the larger angle is opposite the longer side.

Theorem 7-5. (P. 201) If two angles of a triangle are not congruent, then the sides opposite them are not congruent, and the longer side is opposite the larger angle.

Theorem 7-6. (P. 206) The shortest segment joining a point to a line is the perpendicular segment.

Theorem 7-7. (P. 206) (The Triangle Inequality.) The sum of the lengths of any two sides of a triangle is greater than the of the third side.

Theorem 7-8. (P. 210) If two sides of one triangle are congruent respectively to two sides of a second triangle, and the included angle of the first triangle is larger than the included angle of the second, then the opposite side of the first triangle is longer than the opposite side of the second.

Theorem 7-9. (P. 211) If two sides of one triangle are congruent respectively to two sides of a second triangle, and the third side of the first triangle is longer than the third side of the second, then the included angle of the first triangle is larger than the included angle of the second.

Theorem 8-1. (P. 222) If each of two points of a line is equidistant from two given points, then every point of the line is equidistant from the given points.

Theorem 8-2. (P. 225) If each of three non-collinear points of a plane is equidistant from two points, then every point of the plane is equidistant from these two points.

Theorem 8-3. (P. 226) If a line is perpendicular to each of two intersecting lines at their point of intersection, then it is perpendicular to the plane of these lines.

Theorem 8-4. (P. 230) Through a given point on a given line there passes a plane perpendicular to the line.

Theorem 8-5. (P. 231) If a line and a plane are perpendicular, then the plane contains every line perpendicular to the given line at its point of intersection with the given plane.

Theorem 8-6. (P. 232) Through a given point on a given line there is at most one plane perpendicular to the line.

Theorem 8-7. (P. 232) The perpendicular bisecting plane of a segment is the set of all points equidistant from the endpoints of the segment.

Theorem 8-8. (P. 234) Two lines perpendicular to the same plane are coplanar.

Theorem 8-9. (P. 235) Through a given point there passes one and only one *plane* perpendicular to a given *line*.

Theorem 8-10. (P. 235) Through a given point there passes one and only one *line* perpendicular to a given *plane*.

Theorem 8-11. (P. 235) The shortest segment to a plane from an external point is the perpendicular segment.

Theorem 9-1. (P. 242) Two parallel lines lie in exactly one plane.

Theorem 9-2. (P. 242 Two lines in a plane are parallel if they are both perpendicular to the same line.

Theorem 9-3. (P. 244) Let L be a line, and let P be a point not on L. Then there is at least one line through P, parallel to L.

Theorem 9-4. (P. 246) If two lines are cut by a transversal, and if one pair of alternate interior angles are congruent, then the other pair of alternate interior angles are also congruent.

Theorem 9-5. (P. 246) If two lines are cut by a transversal, and if a pair of alternate interior angles are congruent, then the lines are parallel.

Theorem 9-6. (P. 252) If two lines are cut by a transversal, and if one pair of corresponding angles are congruent, then the other three pairs of corresponding angles have the same property.

Theorem 9-7. (P. 252) If two lines are cut by a transversal, and if a pair of corresponding angles are congruent, then the lines are parallel.

Theorem 9-8. (P. 253) If two parallel lines are cut by a transversal, then then alternate interior angles are congruent.

Theorem 9-9. (P. 254) If two parallel lines are cut by a transversal, each pair of corresponding angles are congruent.

Theorem 9-10. (P. 254) If two parallel lines are cut by a transversal, interior angles on the same side of the transversal are supplementary.

Theorem 9-11. (P. 255) In a plane, two lines parallel to the same line are parallel to each other.

Theorem 9-12. (P. 255) In a plane, if a line is perpendicular to one of two parallel lines it is perpendicular to the other.

Theorem 9-13. (P. 258) The sum of the measures of the angles of a triangle is 180.

Corollary 9-13-1. (P. 259) Given a correspondence between two triangles. If two pairs of corresponding angles are congruent, then the third pair of corresponding angles are also congruent.

Corollary 9-13-2. (P. 260) The acute angles of a right triangle are complementary.

Corollary 9-13-3. (P. 260) For any triangle, the measure of an exterior angle is the sum of the measures of the two remote interior angles.

Theorem 9-14. (P. 265) Either diagonal divides a parallelogram into two congruent triangles.

Theorem 9-15. (P. 265) In a parallelogram, any two opposite sides are congruent.

Corollary 9-15-1. (P. 266) If $L_1 \parallel L_2$ and if P and Q are any two points on L_1, then the distances of P and Q from L_2 are equal.

Theorem 9-16. (P. 266) In a parallelogram, any two opposite angles are congruent.

Theorem 9-17. (P. 266) In a parallelogram, any two consecutive angles are supplementary.

Theorem 9-18. (P. 266) The diagonals of a parallelogram bisect each other.

Theorem 9-19. (P. 266) Given a quadrilateral in which both pairs of opposite sides are congruent. Then the quadrilateral is a parallelogram.

Theorem 9-20. (P. 266) If two sides of a quadrilateral are parallel and congruent, then the quadrilateral is a parallelogram.

Theorem 9-21. (P. 266) If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

Theorem 9-22. (P. 267) The segment between the mid-points of two sides of a triangle is parallel to the third side and half as long as the third side.

Theorem 9-23. (P. 268) If a parallelogram has one right angle, then it has four right angles, and the parallelogram is a rectangle.

Theorem 9-24. (P. 268) In a rhombus, the diagonals are perpendicular to one another.

Theorem 9-25. (P. 268) If the diagonals of a quadrilateral bisect each other and are perpendicular, then the quadrilateral is a rhombus.

Theorem 9-26. (P. 276) If three parallel lines intercept congruent segments on one transversal, then they intercept congruent segments on any other transversal.

Corollary 9-26-1. (P. 277) If three or more parallel lines intercept congruent segments on one transversal, then they intercept congruent segments on any other transversal.

Theorem 9-27. (P. 279) The medians of a triangle are concurrent in a point two-thirds the way from any vertex to the mid-point of the opposite side.

Theorem 10-1. (P. 292) If a plane intersects two parallel planes, then it intersects them in two parallel lines.

Theorem 10-2. (P. 292) If a line is perpendicular to one of two parallel planes it is perpendicular to the other.

Theorem 10-3. (P. 293) Two planes perpendicular to the same line are parallel.

Corollary 10-3-1. (P. 294) If two planes are each parallel to a third plane, they are parallel to each other.

Theorem 10-4. (P. 294) Two lines perpendicular to the same plane are parallel.

Corollary 10-4-1. (P. 294) A plane perpendicular to one of two parallel lines is perpendicular to the other.

Corollary 10-4-2. (P. 294) If two lines are each parallel to a third they are parallel to each other.

Theorem 10-5. (P. 295) Two parallel planes are everywhere equidistant.

Theorem 10-6. (P. 301) Any two plane angles of a given dihedral angle are congruent.

Corollary 10-6-1. (P. 302) If a line is perpendicular to a plane, then any plane containing this line is perpendicular to the given plane.

Corollary 10-6-2. (P. 302) If two planes are perpendicular, then any line in one of them perpendicular to their line of intersection is perpendicular to the other plane.

Theorem 10-7. (P. 307) The projection of a line into a plane is a line, unless the line and the plane are perpendicular.

Theorem 11-1. (P. 328) The area of a right triangle is half the product of its legs.

Theorem 11-2. (P. 328) The area of a triangle is half the product of any base and the altitude to that base.

Theorem 11-3. (P. 330) The area of a parallelogram is the product of any base and the corresponding altitude.

Theorem 11-4. (P. 331) The area of a trapezoid is half the product of its altitude and the sum of its bases.

Theorem 11-5. (P. 332) If two triangles have the same altitude, then the ratio of their areas is equal to the ratio of their bases.

Theorem 11-6. (P. 332) If two triangles have equal altitudes and equal bases, then they have equal areas.

Theorem 11-7. (P. 339) (The Pythagorean Theorem.) In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs.

Theorem 11-8. (P. 340) If the square of one side of a triangle is equal to the sum of the squares of the other two, then the triangle is a right triangle, with a right angle opposite the first side.

Theorem 11-9. (P. 346) (The 30 - 60 Triangle Theorem.) The hypotenuse of a right triangle is twice as long as the shorter leg if and only if the acute angles are 30° and 60°.

Theorem 11-10. (P. 346) (The Isosceles Right Triangle Theorem.) A right triangle is isosceles if and only if the hypotenuse is $\sqrt{2}$ times as long as a leg.

Theorem 12-1. (P. 368) (The Basic Proportionality Theorem.) If a line parallel to one side of a triangle intersects the other two sides in distinct points, then it cuts off segments which are proportional to these sides.

Theorem 12-2. (P. 369) If a line intersects two sides of a triangle, and cuts off segments proportional to these two sides, then it is parallel to the third side.

Theorem 12-3. (P. 374) (The A.A.A. Similarity Theorem.) Given a correspondence between two triangles. If corresponding angles are congruent, then the correspondence is a similarity.

Corollary 12-3-1. (P. 376) (The A.A. Corollary.) Given a correspondence between two triangles. If two pairs of corresponding angles are congruent, then the correspondence is a similarity.

Corollary 12-3-2. (P. 376) If a line parallel to one side of a triangle intersects the other two sides in distinct points, then it cuts off a triangle similar to the given triangle.

Theorem 12-4. (P. 376) (The S.A.S. Similarity Theorem.) Given a correspondence between two triangles. If two corresponding angles are congruent, and the including sides are proportional, then the correspondence is a similarity.

Theorem 12-5. (P. 378) (The S.S.S. Similarity Theorem.) Given a correspondence between two triangles. If corresponding sides are proportional, then the correspondence is a similarity.

Theorem 12-6. (P. 391) In any right triangle, the altitude to the hypotenuse separates the triangle into two triangles which are similar both to each other and to the original triangle.

Corollary 12-6-1. (P. 392) Given a right triangle and the altitude from the right angle to the hypotenuse:

(1) The altitude is the geometric mean of the segments into which it separates the hypotenuse.

(2) Either leg is the geometric mean of the hypotenuse and the segment of the hypotenuse adjacent to the leg.

Theorem 12-7. (P. 395) The ratio of the areas of two similar triangles is the square of the ratio of any two corresponding sides.

Theorem 13-1. (P. 410) The intersection of a sphere with a plane through its center is a circle with the same center and radius.

Theorem 13-2. (P. 414) Given a line and a circle in the same plane. Let P be the center of the circle, and let F be the foot of the perpendicular from P to the line. Then either

(1) Every point of the line is outside the circle, or

(2) F is on the circle, and the line is tangent to the circle at F, or

(3) F is inside the circle, and the line intersects the circle in exactly two points, which are equidistant from F.

Corollary 13-2-1. (P. 416) Every line tangent to C is perpendicular to the radius drawn to the point of contact.

Corollary 13-2-2. (P. 416) Any line in E, perpendicular to a radius at its outer end, is tangent to the circle.

Corollary 13-2-3. (P. 416) Any perpendicular from the center of C to a chord bisects the chord.

Corollary 13-2-4. (P. 416) The segment joining the center of C to the mid-point of a chord is perpendicular to the chord.

Corollary 13-2-5. (P. 416) In the plane of a circle, the perpendicular bisector of a chord passes through the center of the circle.

Corollary 13-2-6. (P. 417) If a line in the plane of a circle intersects the interior of the circle, then it intersects the circle in exactly two points.

Theorem 13-3. (P. 417) In the same circle or in congruent circles, chords equidistant from the center are congruent.

Theorem 13-4. (P. 417) In the same circle or in congruent circles, any two congruent chords are equidistant from the center.

Theorem 13-5. (P. 424) Given a plane E and a sphere S with center P. Let F be the foot of the perpendicular segment from P to E. Then either

(1) Every point of E is outside S, or

(2) F is on S, and E is tangent to S at F, or

(3) F is inside S, and E intersects S in a circle with center F.

Corollary 13-5-1. (P. 426) A plane tangent to S is perpendicular to the radius drawn to the point of contact.

Corollary 13-5-2. (P. 426) A plane perpendicular to a radius at its outer end is tangent to S.

Corollary 13-5-3. (P. 426) A perpendicular from P to a chord of S bisects the chord.

Corollary 13-5-4. (P. 426) The segment joining the center of S to the midpoint of a chord is perpendicular to the chord.

Theorem 13-6. (P. 431) If $\overset{\frown}{AB}$ and $\overset{\frown}{BC}$ are arcs of the same circle having only the point B in common, and if their union is an arc $\overset{\frown}{AC}$, then $m\overset{\frown}{AB} + m\overset{\frown}{BC} = m\overset{\frown}{AC}$.

Theorem 13-7. (P. 434) The measure of an inscribed angle is half the measure of its intercepted arc.

Corollary 13-7-1. (P. 437) An angle inscribed in a semicircle is a right angle.

Corollary 13-7-2. (P. 437) Angles inscribed in the same arc are congruent.

Theorem 13-8. (P. 441) In the same circle or in congruent circles, if two chords are congruent, then so also are the corresponding minor arcs.

Theorem 13-9. (P. 441) In the same circle or in congruent circles, if two arcs are congruent, then so are the corresponding chords.

Theorem 13-10. (P. 442) Given an angle with vertex on the circle formed by a secant ray and a tangent ray. The measure of the angle is half the measure of the intercepted arc.

Theorem 13-11. (P. 448) The two tangent segments to a circle from an external point are congruent, and form congruent angles with the line joining the external point to the center of the circle.

Theorem 13-12. (P. 449) Given a circle C and an external point Q, let L_1 be a secant line through Q, intersecting C in points R and S; and let L_2 be another secant line through Q, intersecting C in points T and U. Then $QR \cdot QS = QU \cdot QT$.

Theorem 13-13. (P. 450) Given a tangent segment $\overline{QT}$ to a circle, and a secant line through Q, intersecting the circle in points R and S. Then $QR \cdot QS = QT^2$.

Theorem 13-14. (P. 451) If two chords intersect within a circle, the product of the lengths of the segments of one equals the product of the lengths of the segments of the other.

Theorem 14-1. (P. 467) The bisector of an angle, minus its end-point, is the set of points in the interior of the angle equidistant from the sides of the angle.

Theorem 14-2. (P. 469) The perpendicular bisectors of the sides of a triangle are concurrent in a point equidistant from the three vertices of the triangle.

Corollary 14-2-1. (P. 470) There is one and only one circle through three non-collinear points.

Corollary 14-2-2. (P. 470) Two distinct circles can intersect in at most two points.

Theorem 14-3. (P. 470) The three altitudes of a triangle are concurrent.

Theorem 14-4. (P. 471) The angle bisectors of a triangle are concurrent in a point equidistant from the three sides.

Theorem 14-5. (P. 476) (The Two Circle Theorem) If two circles have radii a and b, and if c is the distance between their centers, then the circles intersect in two points, one on each side of the line of centers, provided each of of a, b, c is less than the sum of the other two.

Construction 14-6. (P. 477) To copy a given triangle.

Construction 14-7. (P. 479) To copy a given angle.

Construction 14-8. (P. 481) To construct the perpendicular bisector of a given segment.

Corollary 14-8-1. (P. 481) To bisect a given segment.

Construction 14-9. (P. 482) To construct a perpendicular to a given line through a given point.

Construction 14-10. (P. 484) To construct a parallel to a given line, through a given external point.

Construction 14-11. (P. 484) To divide a segment into a given number of congruent segments.

Construction 14-12. (P. 491) To circumscribe a circle about a given triangle.

Construction 14-13. (P. 491) To bisect a given angle.

Construction 14-14. (P. 492) To inscribe a circle in a given triangle.

Theorem 15-1. (P. 517) The ratio $\frac{C}{2r}$, of the circumference to the diameter, is the same for all circles.

Theorem 15-2. (P. 522) The area of a circle of radius r is πr^2 .

Theorem 15-3. (P. 526) If two arcs have equal radii, their lengths are proportional to their measures.

Theorem 15-4. (P. 526) An arc of measure q and radius r has length $\frac{\pi}{180} qr$.

Theorem 15-5. (P. 527) The area of a sector is half the product of its radius by the length of its arc.

Theorem 15-6. (P. 527) The area of a sector of radius r and arc measure q is $\frac{\pi}{360} qr^2$.

Theorem 16-1. (P. 535) All cross-sections of a triangular prism are congruent to the base.

Corollary 16-1-1. (P. 536) The upper and lower bases of a triangular prism are congruent.

Theorem 16-2. (P. 536) (Prism Cross-Section Theorem.) All cross-sections of a prism have the same area.

Corollary 16-2-1. (P. 537) The two bases of a prism have equal areas.

Theorem 16-3. (P. 537) The lateral faces of a prism are parallelogram regions, and the lateral faces of a right prism are rectangular regions.

Theorem 16-4. (P. 540) A cross-section of a triangular pyramid, by a plane between the vertex and the base, is a triangular region similar to the base. If the distance from the vertex to the cross-section plane is k and the altitude is h, then the ratio of the area of the cross-section to the area of the base is $(\frac{k}{h})^2$.

Theorem 16-5. (P. 542) In any pyramid, the ratio of the area of a cross-section and the area of the base is $(\frac{k}{h})^2$, where h is the altitude of the pyramid and k is the distance from the vertex to the plane of the cross-section.

Theorem 16-6. (P. 543) (The Pyramid Cross-Section Theorem.) Given two pyramids with the same altitude. If the bases have the same area, then cross-sections equidistant from the bases also have the same area.

Theorem 16-7. (P. 548) The volume of any prism is the product of the altitude and the area of the base.

Theorem 16-8. (P. 549) If two pyramids have the same altitude and the same base area, then they have the same volume.

Theorem 16-9. (P. 550) The volume of a triangular pyramid is one-third the product of its altitude and its base area.

Theorem 16-10. (P. 551) The volume of a pyramid is one-third the product of its altitude and its base area.

Theorem 16-11. (P. 555) A cross-section of a circular cylinder is a circular region congruent to the base.

Theorem 16-12. (P. 555) The area of a cross-section of a circular cylinder is equal to the area of the base.

Theorem 16-13. (P. 555) A cross-section of a cone of altitude h, made by a plane at a distance k from the vertex, is a circular region whose area has a ratio to the area of the base of $(\frac{k}{h})^2$.

Theorem 16-14. (P. 557) The volume of a circular cylinder is the product of the altitude and the area of the base.

Theorem 16-15. (P. 557) The volume of a circular cone is one-third the product of the altitude and the area of the base.

Theorem 16-16. (P. 559) The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

Theorem 16-17. (P. 562) The surface area of a sphere of radius r is $S = 4\pi r^2$.

Theorem 17-1. (P. 579) On a non-vertical line, all segments have the same slope.

Theorem 17-2. (P. 584) Two non-vertical lines are parallel if and only if they have the same slope.

Theorem 17-3. (P. 586) Two non-vertical lines are perpendicular if and only if their slopes are the negative reciprocals of each other.

Theorem 17-4. (P. 589) (The Distance Formula.) The distance between the points (x_1,y_1) and (x_2,y_2) is equal to $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Theorem 17-5. (P. 593) (The Mid-Point Formula.) Let $P_1 = (x_1,y_1)$ and let $P_2 = (x_2,y_2)$. Then the mid-point of $\overline{P_1P_2}$ is the point $P = (\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$

Theorem 17-6. (P. 605) Let L be a non-vertical line with with slope m, and let P be a point of L, with coordinates (x_1,y_1). For every point $Q = (x,y)$ of L, the equation $y - y_1 = m(x - x_1)$ is satisfied.

Theorem 17-7. (P. 607) The graph of the equation $y - y_1 = m(x - x_1)$ is the line that passes through the point (x_1,y_1) and has slope m.

Theorem 17-8. (P. 611) The graph of the equation $y = mx + b$ is the line with slope m and y-intercept b.

Theorem 17-9. (P. 613) Every line in the plane is the graph of a linear equation in x and y.

Theorem 17-10. (P. 613) The graph of a linear equation in x and y is always a line.

Theorem 17-11. (P. 623) The graph of the equation $(x - a)^2 + (y - b)^2 = r^2$ is the circle with center at (a,b) and radius r.

Theorem 17-12. (P. 624) Every circle is the graph of an equation of the form $x^2 + y^2 + Ax + By + C = 0$.

Theorem 17-13. (P. 625) Given the equation $x^2 + y^2 + Ax + By + C = 0$. The graph of this equation is (1) a circle, (2) a point or (3) the empty set.

Index of Definitions

For precisely defined geometric terms the reference is to the formal definition. For other terms the reference is to an informal definition or to the most prominent discussion.

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